ON METRIC INDEPENDENCE AND LINEAR INDEPENDENCE

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For a metric space M, CM will denote the Banach space of all bounded real-valued continuous functions on M, with the usual $\|f\| = \sup_{x\in M}|f(x)|$. It is well-known that M is homeomorphic (and, in fact, isometric) with a subset of CM [4, p. 543]. We show here that M must be homeomorphic with a *linearly independent* subset of CM. Whether M must be isometric with such a set remains undecided.

Let the distance between two points x and y of M be denoted by xy, and for each $x\in M$ let f_x be the function $xy|y\in M$. The subset A of M is said to be *metrically dependent* in M provided the family of functions $\{f_a: a\in A\}$ is linearly dependent over M. Otherwise, A is *metrically independent* in M. This notion leads quickly to the desired result, by means of the following observations.

(1) *If A is a subset of a (not necessarily separable) Hilbert space E, then A is metrically independent in A.*

Proof. It suffices to show that if $p_1, \cdots, p_n$ are n distinct points of A and d is the value of the determinant $|p_ip_j|_{1,n}$, then $d\neq 0$. But since the subspace of E determined by $\{p_1, \cdots, p_n\}$ is a Euclidean space of dimension $\leqq n$, this follows at once by an argument of Schoenberg [5, p. 792]. (See also [2, §40 and §54].)

(2) *If M is a metric space, then M has a bounded homeomorph in which every subset A is metrically independent in A.*

Proof. By a theorem of A. H. Stone [6], M is paracompact, and hence by a theorem of C. H. Dowker [3, p. 639] there is a Hilbert space whose unit sphere contains a homeomorph of M. The desired conclusion then follows from (1).

(3) *If M is a metric space, then M is homeomorphic with a linearly independent subset of CM.*

Proof. In view of (2), we may assume that M is bounded and that every subset A of M is metrically independent in A. Now let $TD = f_x|x\in M$, where f_x is as defined above. From the fact that M is bounded and from the definition of metric independence it follows

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that TM is a linearly independent subset of CM. Since T is an isometry, the proof is complete.

Now if (in the proof of (3)) TM is separable, then so is the linear subspace of CM spanned by TM. From a well-known embedding theorem of Banach and Mazur [1, p. 185] there follows

(4) *Every separable metric space is homeomorphic with a linearly independent subset of* $C[0, 1]$.

We do not know[3] whether every separable metric space is *isometric* with a linearly independent subset of $C[0, 1]$, although it is easy to see that every finite metric space has this property.

Now for an arbitrary metric space M, let K be the unit cell $\{f: \|f\| \leqq 1\}$ of the space $(CM)^*$ dual to CM. Then K is compact in the weak topology, and there is a natural linear isometry of CM into CK. Thus we have

(5) *For each metric space M there is a compact Hausdorff space K such that M is homeomorphic with a linearly independent subset of CK.*

We conclude with some remarks on metric dependence.

(6) *If a metric space M has fewer than four points, then M is metrically independent in M.*

A metric quadruple is *pseudo-linear* [2, p. 110] provided each of its triples is linear (i.e., isometric with a subset of E^1), but the quadruple itself is not linear.

(7) *For a metric quadruple Q, the following three assertions are equivalent*:

(i) *Q is pseudo-linear.*

(ii) *The points of Q can be so labelled that $q_1q_2 = q_3q_4 = a > 0$, $q_2q_3 = q_1q_4 = b > 0$, and $q_1q_3 = q_2q_4 = a + b$.*

(iii) *Q is metrically dependent in Q.*

Proof. That (i) is equivalent to (ii) is noted in [2, p. 114]. From (ii) it follows easily that the determinant $|q_iq_j|_{1,4}$ is zero, whence (ii) implies (iii). That (iii) implies (i) can be proved by applying results of [2] (in particular, pp. 131 and 293) to the metric transform of Q by the function $\phi(x) = x^{1/2}$, but we give here a more elementary proof.

Suppose Q is metrically dependent in Q; i.e., $\sum_1^4 a_i f_{q_i} \equiv 0$ for numbers a_i not all zero. Since, by (1), Q is not linear, to show that Q is pseudo-linear it suffices to show that each triple in Q is linear. It is clear that no a_i can be different in sign from all the other three, for f_{q_i} vanishes only at q_i. Thus with notation appropriately chosen we

[3] *Added in proof*: An affirmative answer to this question follows from a recent theorem of Arens, to the effect that every metric space is isometric with a closed linearly independent subset of some normed linear space.

have $Q=\{w, x, y, z\}$ and positive numbers a, b, c, and d such that (α) $af_w+bf_x=cf_y+df_z$. Evaluating (α) at w and then at x, adding the results and using the triangle inequality, we obtain (β) $(a+b)xw =c(xy+yw)+d(xz+zw)\geqq(c+d)xw$. And similarly (evaluating (α) at y and then at z), (γ) $(c+d)yz=a(yw+wz)+b(yx+xz)\geqq(a+b)yz$. But ($\beta$) implies $c+d\leqq a+b$ and (γ) implies $a+b\leqq c+d$, so $a+b=c+d$ and the inequalities in (β) and (γ) can be replaced by equalities, whence we see at once that each triple in Q is linear.

An easy corollary of (7) is

(8) *A quadruple Q in a metric space M is metrically dependent in M if and only if Q is pseudo-linear and $pq_1+pq_3=pq_2+pq_4$ for each $p\in M$ (where the labelling is as in* (ii) *of* (7)).

References

1. Stefan Banach, *Théorie des operations linéaires*, Warsaw, 1932.

2. Leonard M. Blumenthal, *Theory and applications of distance geometry*, Oxford, 1953.

3. C. H. Dowker, *An imbedding theorem for paracompact metric spaces*, Duke Math. J. vol. 14 (1947) pp. 639–645.

4. C. Kuratowski, *Quelques problemes concernant les espaces métriques non-separables*, Fund. Math. vol. 25 (1935) pp. 534–545.

5. I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. of Math. vol. 38 (1937) pp. 787–793.

6. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 977–982.

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