

ON METRIC INDEPENDENCE AND LINEAR INDEPENDENCE

L. M. BLUMENTHAL¹ AND V. L. KLEE²

For a metric space M , CM will denote the Banach space of all bounded real-valued continuous functions on M , with the usual $\|f\| = \sup_{x \in M} |f(x)|$. It is well-known that M is homeomorphic (and, in fact, isometric) with a subset of CM [4, p. 543]. We show here that M must be homeomorphic with a *linearly independent* subset of CM . Whether M must be isometric with such a set remains undecided.

Let the distance between two points x and y of M be denoted by xy , and for each $x \in M$ let f_x be the function $xy|y \in M$. The subset A of M is said to be *metrically dependent* in M provided the family of functions $\{f_a : a \in A\}$ is linearly dependent over M . Otherwise, A is *metrically independent* in M . This notion leads quickly to the desired result, by means of the following observations.

(1) *If A is a subset of a (not necessarily separable) Hilbert space E , then A is metrically independent in A .*

PROOF. It suffices to show that if p_1, \dots, p_n are n distinct points of A and d is the value of the determinant $|p_i p_j|_{1,n}$, then $d \neq 0$. But since the subspace of E determined by $\{p_1, \dots, p_n\}$ is a Euclidean space of dimension $\leq n$, this follows at once by an argument of Schoenberg [5, p. 792]. (See also [2, §40 and §54].)

(2) *If M is a metric space, then M has a bounded homeomorph in which every subset A is metrically independent in A .*

PROOF. By a theorem of A. H. Stone [6], M is paracompact, and hence by a theorem of C. H. Dowker [3, p. 639] there is a Hilbert space whose unit sphere contains a homeomorph of M . The desired conclusion then follows from (1).

(3) *If M is a metric space, then M is homeomorphic with a linearly independent subset of CM .*

PROOF. In view of (2), we may assume that M is bounded and that every subset A of M is metrically independent in A . Now let $TD = f_x|x \in M$, where f_x is as defined above. From the fact that M is bounded and from the definition of metric independence it follows

Presented to the Society, February 26, 1955; received by the editors December 9, 1954.

¹ Fulbright Lecturer, University of Leiden, The Netherlands.

² Work on this paper sponsored by the Office of Ordnance Research, U. S. Army, under Contract DA-04-200-ORD-292.

that TM is a linearly independent subset of CM . Since T is an isometry, the proof is complete.

Now if (in the proof of (3)) TM is separable, then so is the linear subspace of CM spanned by TM . From a well-known embedding theorem of Banach and Mazur [1, p. 185] there follows

(4) *Every separable metric space is homeomorphic with a linearly independent subset of $C [0, 1]$.*

We do not know³ whether every separable metric space is *isometric* with a linearly independent subset of $C [0, 1]$, although it is easy to see that every finite metric space has this property.

Now for an arbitrary metric space M , let K be the unit cell $\{f: \|f\| \leq 1\}$ of the space $(CM)^*$ dual to CM . Then K is compact in the weak topology, and there is a natural linear isometry of CM into CK . Thus we have

(5) *For each metric space M there is a compact Hausdorff space K such that M is homeomorphic with a linearly independent subset of CK .*

We conclude with some remarks on metric dependence.

(6) *If a metric space M has fewer than four points, then M is metrically independent in M .*

A metric quadruple is *pseudo-linear* [2, p. 110] provided each of its triples is linear (i.e., isometric with a subset of E^1), but the quadruple itself is not linear.

(7) *For a metric quadruple Q , the following three assertions are equivalent:*

- (i) Q is pseudo-linear.
- (ii) The points of Q can be so labelled that $q_1q_2 = q_3q_4 = a > 0$, $q_2q_3 = q_1q_4 = b > 0$, and $q_1q_3 = q_2q_4 = a + b$.
- (iii) Q is metrically dependent in Q .

PROOF. That (i) is equivalent to (ii) is noted in [2, p. 114]. From (ii) it follows easily that the determinant $|q_iq_j|_{1,4}$ is zero, whence (ii) implies (iii). That (iii) implies (i) can be proved by applying results of [2] (in particular, pp. 131 and 293) to the metric transform of Q by the function $\phi(x) = x^{1/2}$, but we give here a more elementary proof.

Suppose Q is metrically dependent in Q ; i.e., $\sum_1^4 a_i f_{q_i} \equiv 0$ for numbers a_i not all zero. Since, by (1), Q is not linear, to show that Q is pseudo-linear it suffices to show that each triple in Q is linear. It is clear that no a_i can be different in sign from all the other three, for f_{q_i} vanishes only at q_i . Thus with notation appropriately chosen we

³ *Added in proof:* An affirmative answer to this question follows from a recent theorem of Arens, to the effect that every metric space is isometric with a closed linearly independent subset of some normed linear space.

have $Q = \{w, x, y, z\}$ and positive numbers a, b, c , and d such that (α) $af_w + bf_x = cf_y + df_z$. Evaluating (α) at w and then at x , adding the results and using the triangle inequality, we obtain (β) $(a+b)xw = c(xy + yw) + d(xz + zw) \geq (c+d)xw$. And similarly (evaluating (α) at y and then at z), (γ) $(c+d)yz = a(yw + wz) + b(yx + xz) \geq (a+b)yz$. But (β) implies $c+d \leq a+b$ and (γ) implies $a+b \leq c+d$, so $a+b = c+d$ and the inequalities in (β) and (γ) can be replaced by equalities, whence we see at once that each triple in Q is linear.

An easy corollary of (7) is

(8) *A quadruple Q in a metric space M is metrically dependent in M if and only if Q is pseudo-linear and $pq_1 + pq_3 = pq_2 + pq_4$ for each $p \in M$ (where the labelling is as in (ii) of (7)).*

REFERENCES

1. Stefan Banach, *Théorie des opérations linéaires*, Warsaw, 1932.
2. Leonard M. Blumenthal, *Theory and applications of distance geometry*, Oxford, 1953.
3. C. H. Dowker, *An imbedding theorem for paracompact metric spaces*, Duke Math. J. vol. 14 (1947) pp. 639–645.
4. C. Kuratowski, *Quelques problèmes concernant les espaces métriques non-séparables*, Fund. Math. vol. 25 (1935) pp. 534–545.
5. I. J. Schoenberg, *On certain metric spaces arising from Euclidean spaces by a change of metric and their imbedding in Hilbert space*, Ann. of Math. vol. 38 (1937) pp. 787–793.
6. A. H. Stone, *Paracompactness and product spaces*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 977–982.

UNIVERSITY OF LEIDEN,
UNIVERSITY OF MISSOURI, AND
UNIVERSITY OF WASHINGTON