THREE POINT ARCWISE CONVEXITY

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Let S be a set in a two dimensional Euclidean space E_2. Such a set S is said to be *arcwise convex* [**5**] if each pair of its points can be joined by a convex arc lying in S. A *convex arc* is, by definition, an arc which is contained in the boundary of a plane convex set. In two previous papers, [**5**; **6**], the author studied certain properties of closed arcwise convex sets. It is the purpose of this paper to study an interesting class of sets which satisfy the *three point arcwise convexity property*, defined as follows:

DEFINITION 1. *A set $S \subset E_2$ is said to have the three point arcwise convexity property if each triple of points $x \in S$, $y \in S$, $z \in S$ is contained in a convex arc belonging to S.*

It should be observed that the above property implies there exists a convex arc in S having two of the three points x, y, z as end points and the third point in its interior.

DEFINITION 2. *A convex curve, as distinguished from a convex arc, is a closed connected portion of the boundary of a plane convex set.*

A convex curve may have two, one, or no end points, and it may be bounded or unbounded.

The following theorem characterizes the closed sets in E_2 which have the three point arcwise convexity property.

THEOREM 1. *Let S be a closed set in E_2 which has at least three points. Then S has the three point arcwise convexity property if and only if it satisfies at least one of the following three conditions.*

1. *It is a closed convex set.*
2. *It is a convex curve.*
3. *It is a closed convex set except for one bounded convex hole, that is, it is obtained by deleting from a closed convex set a bounded open convex subset.*

In order to prove the necessity of Conditions 1–3, we shall establish four lemmas. In each of them it is assumed that S *is a closed set which has at least three points, and which has the three point arcwise convexity property.*

LEMMA 1. *If x, y, z are three collinear points of S, with y between x and z, then at least one of the closed segments xy or yz belongs to S.*

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PROOF. In this and later proofs we use the following notation.

NOTATION. The boundary of a set C is denoted by $B(C)$. A convex arc in S containing the points x, y, and z and having two of them as end points is denoted by $C(x, y, z)$.

To prove Lemma 1, let H be the convex hull of $C(x, y, z)+xz$. Since $C(x, y, z)$ is a convex arc, we have $C\equiv C(x, y, z)\subset B(H)$. This implies that if z is an end point of C, then $xy\subset S$. On the other hand, if z is not an end point of C, then $yz\subset S$.

LEMMA 2. *Suppose a component K of the complement of S exists which is unbounded, and which has a cross-cut [a cross-cut of K is a closed segment xy with $xy-x-y\subset K$ and with $x\in B(K)$, $y\in B(K)$]. Then $S=B(K)$, and $B(K)$ is a convex curve.*

PROOF. Since K is a component of the complement of a closed connected set, it is known [**3**, pp. 108, 117] that $K-xy$ is the sum of two mutually exclusive open connected sets. One of these is bounded, since S has the three point arcwise convexity property. We denote this bounded component by K_1. We shall prove that $\overline{K}_1$ (the closure of K_1) is a convex set. It follows from Lemma 1 immediately that K_1 lies entirely in one of the two open half-planes determined by the line $L(x, y)$ containing xy. Denote this open half-plane by R. Moreover, let $L(x)$ be a closed ray with x as its end point. If $L(x)\cdot K_1\neq 0$, Lemma 1 implies $L(x)\cdot K_1$ is an interval with end points z and p, where $p\neq x$, and $xz\subset xp$. If $z\neq x$, then the convex arc $C(x, y, z)$ would contain z in the interior of its convex hull, since K is unbounded, and since $L(x)\cdot K_1\neq 0$. However, this contradicts the convexity of the arc $C(x, y, z)$, so that we must have $L(x)\cdot B(K_1)=x+p$ when $L(x)\cdot K_1\neq 0$. Similarly, $L(y)\cdot B(K_1)=y+s$, if $L(y)\cdot K_1\neq 0$. Now, choose any pair of points $u\in K_1$, $v\in K_1$. If the line $L(u, v)$ intersects the interior of the segment xy, then the lines $L(x, u)$, $L(y, u)$, $L(x, v)$, $L(y, v)$ determine a quadrilateral Q having its sides in K_1, and having uv as a diagonal. This, together with the connectness of $B(K_1)$, implies that $Q\subset K_1$. Hence, $uv\subset K_1$. Secondly, if $L(u, v)\cdot xy=0$, then Lemma 1 implies that $L(u, v)\cdot K_1$ is connected. Hence, we have shown that K_1 is convex. Thus, $\overline{K}_1$ is also convex.

Next, we prove that $S\cdot R=B(K_1)-xy$, where R is the open half-plane defined above. First, suppose there exists a point $q\in S\cdot R-\overline{K}_1$ such that $L(x, q)\cdot K_1\neq 0$. Then, as proved above, $L(x, q)\cdot B(K_1)=x+p$, and p is between x and q on the line $L(x, q)$. The line $L(x, q)$ determines an open half-plane R_1 which does not contain y. Choose a ray $L(q)$ with end point q which is in $\overline{R}_1$, and which intersects $K_1\cdot R_1$. Since $\overline{K}_1$ is convex, and since $q\notin\overline{K}_1$, we have $L(q)\cdot B(K_1)=u+v$, where $u\in S\cdot R_1$, $v\in S\cdot R_1$, and where u is between q and v on $L(q)$.

Since K is unbounded, a convex arc $C(x, u, q) \subset S$ is contained in $\bar{R}_1$. Since $L(q) \cdot K_1 \neq 0$, and since the convex hull K^* of $C(x, u, q) + xq$ contains $\bar{K}_1 \cdot \bar{R}_1$, the point u is an interior point of K^*, and this is impossible since $C(x, u, q)$ is a convex arc. Hence, we have a contradiction, and therefore no point $q \in S \cdot R - \bar{K}_1$ exists for which $L(x, q) \cdot K_1 \neq 0$. Similarly no point $q \in S \cdot R - \bar{K}_1$ exists for which $L(y, q) \cdot K_1 \neq 0$. Finally, if a point $q \in S \cdot \bar{R} - \bar{K}_1$ exists such that $xq \cdot K_1 = 0$, $yq \cdot K_1 = 0$, then a convex arc $C(x, y, q) \subset S$ bounds a convex set containing K_1 in its interior. However, in this case, there would exist a ray $L(u)$ with $u = x$ or $u = y$ such that $L(u) \cdot K_1 \neq 0$, and such that $L(u) \cdot S \neq L(u) \cdot B(K_1)$. This contradicts the previously proven fact that when $L(u) \cdot K_1 \neq 0$, then $L(u) \cdot S$ consists of two points of S. Thus $S \cdot R = B(K_1) - xy \equiv C$, a convex arc of S. This is true for *each* cross-cut of K.

Now, to complete the proof of Lemma 2, since K has a cross-cut, the above implies that $B(K) \cdot S$ has a nonempty convex arc C as a subset. Let C^* be the maximal closed convex *curve* of $B(K) \cdot S$ which contains C. We shall prove that $C^* = B(K) = S$. To do this, *first* suppose C^* has two end points, x and y, and let R be the open half-plane containing C^* as defined above by means of $L(x, y)$. Suppose a point $z \in S$ exists which is not in $\bar{R}$. Since $S \cdot \bar{R} = C^*$, a convex arc $C(x, y, z) \subset S$ containing x, y and z would contradict the fact that C^* is maximal. Hence, in this case, $S \cdot \bar{R} = C^* = S$. Similarly, if C^* has one or no end points, Lemma 1 and the maximality of C^* imply that $C^* = S$, a convex curve. This completes the proof of Lemma 2.

LEMMA 3. *The complement of the set S has at most one bounded component, and it is convex.*

PROOF. This is an immediate consequence of Lemma 1.

The following elementary lemma is known, and it is included for sake of completeness.

LEMMA 4. *If an open set K has no cross-cuts, then it is the complement of a convex set.*

PROOF. Let x and y be two points not in K. If $xy \cdot K \neq 0$, then it is clear that K has a cross-cut which is a subinterval of xy. Hence K is the complement of a convex set.

PROOF OF THEOREM 1. *Necessity.* If a set S satisfies the conditions of Lemma 2, then it satisfies Condition 2 of the theorem. If K satisfies the conditions of Lemma 4, then S satisfies either Conditions 1 or 3 of the theorem. If the complement of S is bounded, Lemma 3 implies that S satisfies Condition 3. Thus Theorem 1 is a direct consequence of Lemmas 1–4.

Sufficiency. If S satisfies Condition 1 or 2 of Theorem 1, it obviously satisfies the three point convexity property. Suppose S then satisfies Condition 3, and let x, y, z be any three points in S, and let K be the bounded component of the complement. Let H be the convex hull of $\overline{K}+x+y+z$. Since $S+K$ is a convex set, the boundary $B(H)$ is in S. If $x\in B(H)$, $y\in B(H)$, $z\in B(H)$ then clearly x, y, and z are contained in a convex arc of S. If two of the points, x, y and z, say x and y, are in $B(H)$, and if the third is interior to H, then an arc of $B(H)$ joining x and y and a line segment *in* S joining z to x or z to y (one of these must exist) again provide the desired convex arc. Thus S satisfies the three point arcwise convexity property. This completes the proof of Theorem 1.

Otto Haupt in a paper titled *Über eine Kennzeichnung der Kugel* [2], studied sets $S\subset E_n$ such that for each triple of points in S there exists a circular arc or a segment in S which contains the three points. He calls such sets "Kreiskonvex." *If S is a compact kreiskonvex set containing interior points, Haupt proves that S is a sphere.* It is interesting to note that if in Haupt's theorem, we replace circular arcs by convex arcs, and if each three collinear points of S are contained in a segment of S, then Theorem 1 implies that S is a convex set. These last results are related to a result of G. Aumann [1]. "If each plane section of a compact set S is a simply-connected continuum, then S is convex."

E. G. Straus and F. A. Valentine [4] have investigated sets which have the "three-point linear convexity property." Such a set S is one for which each of its triple of collinear points is contained in a segment of S. They have characterized the connected closed sets $S\subset E_2$ having this property, and the results will be prepared for possible publication.

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