DISTRIBUTIVE SUBLATTICES OF A MODULAR LATTICE[1]

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In Theorem 2 we give a necessary and sufficient condition for a subset of a modular lattice to generate a distributive sublattice. In Theorems 4, 5, and 6 we apply this result to certain rather special cases, where the given condition yields particularly simple criteria.

It is well known[2] that three elements x, y, and z of a modular lattice A generate a distributive sublattice of A if and only if $(x+y)z=xz+yz$. Since this fact can be obtained as a corollary of our result (Theorem 3), we refrain from using it in the proof. We shall however need the following

LEMMA 1. *If A is a modular lattice and $x, y, z\in A$, then the following six conditions are equivalent*:

$$(x+y)z = xz + yz, \qquad (y+z)x = yx + zx, \qquad (z+x)y = zy + xy,$$
$$xy + z = (x+z)(y+z), \qquad yz + x = (y+x)(z+x),$$
$$zx + y = (z+y)(x+y).$$

PROOF. If the first equation holds, then

$$(y+x)(z+x) = (y+x)z + x = yz + xz + x = yz + x,$$
$$zy + xy = (zy+x)y = (z+x)(y+x)y = (z+x)y.$$

Thus the first equation implies the fifth and the third. The proof is easily completed by duality, and by permuting x, y, and z.

THEOREM 2. *Suppose X is a nonempty subset of a modular lattice A. In order for the sublattice of A generated by X to be distributive it is necessary and sufficient that*

$$\text{(i)} \qquad \left(\sum_{i=1}^{m} x_i\right)\prod_{j=1}^{n} y_j = \sum_{i=1}^{m}\left(x_i \prod_{j=1}^{n} y_j\right)$$

whenever m and n are positive integers and $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\in X$.

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[2] O. Ore, *On the foundation of abstract algebra*. I, Ann. of Math. vol. 36 (1935) pp. 415–416.

PROOF. This condition is obviously necessary. To prove that it is also sufficient, we first show that any set X which satisfies this condition also satisfies its dual

$$\left(\prod_{i=1}^{m} x_i\right) + \sum_{j=1}^{n} y_j = \prod_{i=1}^{m}\left(x_i + \sum_{j=1}^{n} y_j\right). \tag{1}$$

This equation clearly holds for $m=1$. Assuming that it holds for $m=k$, we consider the case in which $m=k+1$. Then

$$\begin{aligned}
\prod_{i=1}^{m}\left(x_i + \sum_{j=1}^{n} y_j\right) &= \left(x_1 + \sum_{j=1}^{n} y_j\right)\prod_{i=2}^{m}\left(x_i + \sum_{j=1}^{n} y_j\right)\\
&= \left(x_1 + \sum_{j=1}^{n} y_j\right)\left[\left(\prod_{i=2}^{m} x_i\right) + \sum_{j=1}^{n} y_j\right]\\
&= \left(x_1 + \sum_{j=1}^{n} y_j\right)\left(\prod_{i=2}^{m} x_i\right) + \sum_{j=1}^{n} y_j\\
&= \left(\prod_{i=1}^{m} x_i\right) + \sum_{j=1}^{n}\left(y_j \prod_{i=2}^{m} x_i\right) + \sum_{j=1}^{n} y_j\\
&= \left(\prod_{i=1}^{m} x_i\right) + \sum_{j=1}^{n} y_j.
\end{aligned}$$

Our statement now follows by induction.

Now suppose X satisfies the condition of the theorem. Then X is contained in a maximal subset Y of A with the property that (i) holds whenever the elements x_i and y_i belong to Y. We shall prove that Y is a sublattice of A; since Y is obviously distributive, this will complete the proof.

Suppose $u, v \in Y$, and let $Z = Y \cup \{uv\}$. In order to show that (i) holds whenever the elements x_i and y_j belong to Z, we need only consider the case in which $x_1, x_2, \cdots, x_{m-1}, y_1, y_2, \cdots, y_n \in Y$ and $x_m = uv$. Letting

$$x = \sum_{i=1}^{m-1} x_i \quad \text{and} \quad y = \prod_{j=1}^{n} y_j,$$

we infer from (1) that

$$x_m y + x = (u + x)(v + x)\prod_{j=1}^{n}(y_j + x) = (x_m + x)(y + x).$$

It follows by Lemma 1 that

$$(x + x_m)y = xy + x_m y,$$

whence

$$\left(\sum_{i=1}^{m} x_i\right) y = \left(\sum_{i=1}^{m-1} x_i\right) y + x_m y = \left(\sum_{i=1}^{m-1} x_i y\right) + x_m y = \sum_{i=1}^{m} x_i y,$$

as was to be proved. Since $Y \subseteq Z$, we infer from the maximality of Y that $Y = Z$ and hence $uv \in Y$. By a dual argument we see that Y is closed under addition, and the proof is complete.

In order to verify the condition of the above theorem for a given set X, we need only consider the case in which the elements $x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n$ are all distinct. We may also omit from the sequence $y_1, y_2, \cdots, y_n$ $(x_1, x_2, \cdots, x_m)$ any term which contains (is contained in) another term of the sequence. In the applications which follow, we shall frequently make use of these observations.

THEOREM 3. *Suppose A is a modular lattice and $x, y, z \in A$. In order for the sublattice of A generated by the set $\{x, y, z\}$ to be distributive it is necessary and sufficient that*

$$(x + y)z = xy + xz.$$

PROOF. According to Theorem 2, the set $\{x, y, z\}$ generates a distributive sublattice of A if and only if

$$(x + y)z = xz + yz, \quad (y + z)x = yx + zx, \quad \text{and} \quad (z + x)y = zy + xy,$$

but according to Lemma 1 the first of these three equations implies the other two.

THEOREM 4. *Suppose A is a modular lattice and $x, y, z, u \in A$. In order for the sublattice of A generated by the set $\{x, y, z, u\}$ to be distributive it is necessary and sufficient that*

$$(y + z)u = yu + zu, \qquad (z + u)x = zx + ux, \qquad (u + x)y = uy + xy,$$
$$(x + y)z = xz + yz, \qquad (x + y + z)u = xu + yu + zu, \quad \textit{and}$$
$$xyz + u = (x + u)(y + u)(z + u).$$

PROOF. Assume that these six equations hold. According to Theorem 2 we need only verify the equations

$$(y + z)u = yu + zu, \qquad (x + y + z)u = xu + yu + zu,$$
$$(x + y)zu = xzu + yzu,$$

and the equations obtained from these by permuting x, y, z, and u.

Now it follows from Theorem 3 and the first four of our six equations that any three of the four elements x, y, z, and u generate a distributive lattice. Next observe that

$$(y+z+u)x \leqq (y+z+u)(x+y+z) = y+z+u(x+y+z)$$
$$= y+z+ux+uy+uz = y+z+ux,$$

and hence

$$(y+z+u)x = (y+z+ux)x = (y+z)x+ux = yx+zx+ux.$$

Similarly

$$(z+u+x)y = zy+uy+xy \text{ and } (u+x+y)z = uz+xz+yz.$$

Dually we have

$$xzu+x = (y+x)(z+x)(u+x), \quad zux+y = (z+y)(u+y)(x+y),$$
$$uxy+z = (u+z)(x+z)(y+z).$$

Furthermore

$$xzu+yzu = (zux+y)zu = (z+y)(u+y)(x+y)zu = (x+y)zu.$$

Thus $(x+y)zu = xzu+yzu$, and we can similarly prove each of the equations obtained from this one by permuting x, y, z, and u. This completes the proof of the necessity of our conditions; the converse is obvious.

THEOREM 5. *Suppose A is a modular lattice, p is a positive integer, and $X_1, X_2, \cdots, X_p$ are nonempty linearly ordered subsets of A. In order that the sublattice of A generated by the set $X_1 \cup X_2 \cup \cdots \cup X_p$ be distributive it is necessary and sufficient that, for any $x_1 \in X_1$, $x_2 \in X_2, \cdots, x_p \in X_p$, the sublattice of A generated by the set $\{x_1, x_2, \cdots, x_p\}$ be distributive.*

PROOF. Suppose this condition holds. We may assume that A has a zero element and a unit element, and that these two elements belong to each of the sets X_i. In verifying the condition of Theorem 2, we need therefore only consider the case in which $m=n=p$ and $x_i, y_i \in X_i$ for $i=1, 2, \cdots, p$. The alternative case being trivial, we may also assume that $x_i < y_i$ for $i=1, 2, \cdots, p$.

For any nonnegative integers m and n with $n \leqq m \leqq p$, consider the following proposition:

$P_{m,n}$. If I, J and K are subsets of the set $\{1, 2, \cdots, p\}$ such that $K \subseteq I$, I contains exactly m elements, and K contains exactly n elements, then

$$\left[\sum_{i\in I}\left(x_i \prod_{j\in J} y_j\right)\right]\prod_{j\in K} y_j = \sum_{i\in I}\left(x_i \prod_{j\in J\cup K} y_j\right).$$

This proposition is trivially true if $n=0$ and m is any one of the integers 0, 1, $\cdots$, p, as well as in case $m=n=1$. Now suppose m is one of the integers 2, 3, $\cdots$, p and n is one of the integers 1, 2, $\cdots$, m, and assume that $\mathrm{P}_{m-1,k}$ holds for $k=1, 2, \cdots, m-1$ and that $\mathrm{P}_{m,n-1}$ holds. Assuming that I, J, and K are subsets of the set $\{1, 2, \cdots, p\}$ such that $K\subseteq I$, I has exactly m elements, and K has exactly n elements, we choose $k\in K$ and let

$$I_0 = I - \{k\}, \qquad K_0 = K - \{k\}, \qquad J_0 = J \cup \{k\},$$
$$J_1 = J \cap I_0, \qquad J_2 = J - J_1, \qquad J_3 = J_2 \cup \{k\}.$$

We then have

$$\begin{aligned}
&\Big[\sum_{i\in I}\Big(x_i \prod_{j\in J} y_j\Big)\Big]\prod_{j\in K} y_j \\
&\qquad = \Big[\Big(x_k \prod_{j\in J} y_j\Big) + \sum_{i\in I_0}\Big(x_i \prod_{j\in J} y_j\Big)\Big] y_k \prod_{j\in K} y_j \\
&\qquad = \Big[\Big(x_k \prod_{j\in J} y_j\Big) + \Big[\sum_{i\in I_0}\Big(x_i \prod_{j\in J} y_j\Big)\Big] y_k\Big]\prod_{j\in K_0} y_j.
\end{aligned}$$

Using $\mathrm{P}_{m-1,k}$ where k is the number of elements in J_1, as well as the fact that the set $\{x_i \mid i\in I_0\}\cup\{y_j \mid j\in J_3\}$ generates a distributive sublattice of A, we see that

$$\begin{aligned}
\Big[\sum_{i\in I_0}\Big(x_i \prod_{j\in J} y_j\Big)\Big] y_k &= \Big[\sum_{i\in I_0}\Big(x_i \prod_{j\in J_2} y_j\Big)\Big] y_k \prod_{j\in J_1} y_j \\
&= \Big[\sum_{i\in I_0}\Big(x_i \prod_{j\in J_3} y_j\Big)\Big]\prod_{j\in J_1} y_j \\
&= \sum_{i\in I_0}\Big(x_i \prod_{j\in J_0} y_j\Big).
\end{aligned}$$

We finally use $\mathrm{P}_{m,n-1}$ and the fact that $x_k < y_k$ to conclude that

$$\begin{aligned}
&\Big[\sum_{i\in I}\Big(x_i \prod_{j\in J} y_j\Big)\Big]\prod_{j\in K} y_j \\
&\qquad = \Big[\Big(x_k \prod_{j\in J_0} y_j\Big) + \sum_{i\in I_0}\Big(x_i \prod_{j\in J_0} y_j\Big)\Big]\prod_{j\in K_0} y_j \\
&\qquad = \Big[\sum_{i\in I}\Big(x_i \prod_{j\in J_0} y_j\Big)\Big]\prod_{j\in K_0} y_j = \sum_{i\in I}\Big(x_i \prod_{j\in J\cup K} y_j\Big).
\end{aligned}$$

Hence $\mathrm{P}_{m,n}$ holds in this case, and it follows by an easy induction that $\mathrm{P}_{m,n}$ holds whenever m and n are non-negative integers with

$n \leqq m \leqq p$. Taking $m = n = p$, we infer that

$$\left(\sum_{i=I}^{p} x_i\right) \prod_{j=1}^{p} y_j = \sum_{i=1}^{p} \left(x_i \prod_{j=1}^{p} y_j\right),$$

as was to be proved. Thus the necessity of our conditions follows from Theorem 2; the converse is obvious.

By applying Theorem 5 with $p = 2$ we obtain the well-known fact[3] that the union of two linearly ordered subsets of a modular lattice always generates a distributive sublattice. This special case of Theorem 5 could of course be derived directly from Theorem 2 without going through the inductive argument used in establishing the more general result.

THEOREM 6. *Suppose A is a modular lattice and B and C are distributive sublattices of A. In order for the sublattice of A generated by the set $B \cup C$ to be distributive it is necessary and sufficient that*

$$(b_1 + b_2)c = b_1 c + b_2 c \quad \textit{and} \quad (c_1 + c_2)b = c_1 b + c_2 b$$

whenever

$$b, b_1, b_2 \in B \quad \textit{and} \quad c, c_1, c_2 \in C.$$

PROOF. According to Theorem 3 the above condition implies that any three elements of the set $B \cup C$ generate a distributive sublattice of A. By induction we see that

$$\left(\sum_{i=1}^{m} b_i\right) c = \sum_{i=1}^{m} (b_i c) \quad \text{and} \quad \left(\sum_{i=1}^{m} c_i\right) b = \sum_{i=1}^{m} (c_i b)$$

whenever m is a positive integer, $b, b_1, b_2, \cdots, b_m \in B$, and $c, c_1, c_2, \cdots, c_m \in C$.

We may assume that A has a zero element and a unit element, and that these two elements belong to both B and C. In showing that the set $X = B \cup C$ satisfies the condition of Theorem 2 we need only consider the case in which the first r (s) terms of the sequence $x_1, x_2, \cdots, x_m$ ($y_1, y_2, \cdots, y_n$) belong to B, while the remaining terms belong to C. Letting

$$b_1 = \sum_{i=1}^{r} x_i, \quad c_1 = \sum_{i=r+1}^{m} x_i, \quad b_2 = \prod_{j=1}^{s} y_j, \quad \text{and} \quad c_2 = \prod_{j=s+1}^{m} y_j,$$

we then have $b_1, b_2 \in B$ and $c_1, c_2 \in C$. Hence

[3] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Colloquium Publications, vol. 25, rev. ed., New York, 1948, p. 72.

$$\left(\sum_{i=1}^{m} x_i\right)\prod_{j=1}^{n} y_j = (b_1 + c_1)b_2c_2 = (b_1b_2 + c_1b_2)c_2 = (b_1b_2 + c_1)\, b_2c_2$$

$$= (b_1b_2c_2 + c_1c_2)b_2 = b_1b_2c_2 + c_1b_2c_2$$

$$= \left(\sum_{i=1}^{r} x_i\right) b_2c_2 + \left(\sum_{i=r+1}^{m} x_i\right) b_2c_2$$

$$= \left(\sum_{i=1}^{r} x_ib_2\right) c_2 + \left(\sum_{i=r+1}^{m} x_ic_2\right) b_2$$

$$= \left(\sum_{i=1}^{r} x_ib_2c_2\right) + \left(\sum_{i=r+1}^{m} x_ib_2c_2\right) = \sum_{i=1}^{m}\left(x_i \prod_{j=1}^{n} y_j\right).$$

Thus the necessity of our conditions follows from Theorem 2; the converse is obvious.

In conclusion we remark that the criterion for distributivity given in Theorem 2 cannot be replaced by the simpler condition that

$$\left(\sum_{i=1}^{m} x_i\right) y = \sum_{i=1}^{m} x_iy$$

whenever m is a positive integer and $x_1, x_2, \cdots, x_m, y \in X$. To see this we let A be the lattice of all subspaces of a projective plane, and choose for X a set consisting of four lines, no three of which are concurrent. However, it is not known whether this condition together with its dual is sufficient to insure the distributivity of the sublattice of A generated by X.

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