ON CERTAIN SUBSETS OF FINITE BOOLEAN ALGEBRAS

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1. The boolean algebra B_n, of finite dimension n, may be written as the direct union $B_1 \times B_1 \times \cdots \times B_1$ of n copies of B_1. Consequently each element u of B_n may be represented by an n-digit binary number. Let G_n be the group of those permutations on the elements of B_n which interchange or invert various of the factors in the direct union expansion. Thus the elements of G_n permute the components of the u or interchange 0, 1 in certain components of every u in B_n. The order of G_n is therefore $2^n n!$.

Two subsets of B_n will be called *congruent modulo* G_n if one is the image of the other under transformation by an element of G_n. Clearly sets congruent modulo G_n have the same number of elements. The number of elements in a subset will be called the *order* of the subset. Let $N_n^{(s)}$ be the number of congruence classes of subsets of order s. Note that $N_n^{(s)} = N_n^{(2^n - s)}$. Pólya [1] has calculated $N_n^{(s)}$ for $0 \leqq s \leqq 2^n$ and $n = 1, 2, 3, 4$, and Slepian [2] has found the values of $N_n = \sum_{s=0}^{2^n} N_n^{(s)}$ for $n = 5$, 6. Trivially, $N_n^{(0)} = N_n^{(1)} = 1$, all n; and it is almost as obvious that $N_n^{(2)} = n$, all $n \geqq 1$ (see §2).

In this note an elementary argument is given which yields an explicit expression for $N_n^{(3)}$ good for all $n > 1$ (Theorem 2).

2. The procedure for calculating $N_n^{(3)}$ is based on the notion of the "dimension" of a subset of B_n. Let S be a subset of B_n whose order is at least 2. Let $\vee S$ and $\wedge S$ be the lattice-union and lattice-intersection, respectively, of the elements of S. (The symbols $\vee$, $\wedge$, and $\leqq$ will be used for the lattice operations and the ordering relation in B_n, while $\cup$, $\cap$, and $\subseteq$ will be reserved for their set-theoretical counterparts.) Since the quotient $\vee S / \wedge S$ is relatively complemented, it is a boolean algebra, say B_r if of dimension r. B_r may be called the *connected closure* of S. The relation between a subset S and its connected closure will be written: $S \prec B_r$. The *dimension* of S is defined to be the dimension of B_r if $S \prec B_r$. Thus $S \prec B_r$ implies $\dim S = r$. If the order of S is two, $\dim S$ is simply the usual metric in B_n.

The technique to be used for counting incongruent sets in B_n is to determine the number of incongruent sets of given order having maximal dimension in B_k for each $k \leqq n$. Thus, for instance, $N_n^{(2)} = n$,

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all $n \geqq 1$; for in B_k all sets of order two and maximal dimension are congruent.

LEMMA. *Suppose S is a subset of B_n of order 3 or more. If $S=\{u\}\cup T$ and $T \prec B_r$, then S has maximal dimension if and only if $u \in B_r'$ where B_r' is the set of the complements of the elements of B_r.*

PROOF. If $u \in B_r'$, then $u \wedge (\wedge T) \leqq u \wedge u' = 0$ and $u \vee (\vee T) \geqq u \vee u' = I$, i.e. $S=\{u\}\cup T$ has maximal dimension. But if $u \notin B_r'$, and $v \in B_r$ then either $u \wedge v > 0$ or $u \vee v < I$, since complements are unique, and in this case dim S cannot be maximal.

3. Consider now the case $s=3$ and suppose $S=\{u_1, u_2, u_3\}$. Let $r_i = \dim \{u_j, u_k\}$, where i, j, k is some permutation of 1, 2, 3. Clearly $1 \leqq r_i \leqq n$.

THEOREM 1. dim $\{u_1, u_2, u_3\} = n$ *if and only if* $r_1+r_2+r_3=2n$.

PROOF. If $u_i = x_1^i x_2^i \cdots x_n^i$, $1 \leqq i \leqq 3$, are the binary representations of the u_i, no generality is lost by assuming $x_i^1 = 1$, $1 \leqq i \leqq n$; $x_i^2=1$, $1 \leqq i \leqq n-r_3$; $x_i^2 = 0$, $n-r_3+1 \leqq i \leqq n$. Suppose further that $x_i^3=0$, $1 \leqq i \leqq k$ and $n-r_3+1 \leqq i \leqq n-l$; $x_i^3=1$, $k+1 \leqq i \leqq n-r_3$ and $n-l+1 \leqq i \leqq n$; where $k \leqq n-r_3$, $l \leqq r_3$. The lemma implies that dim $\{u_1, u_2, u_3\} = n$ if and only if $k=n-r_3$. But $r_1=k+l$, $r_2=k+r_3-l$, so that $r_1+r_2+r_3=2(k+r_3)$; from which the theorem follows.

The expressions for the binary components also give the following immediate corollary.

COROLLARY. *If $\{u_1, u_2, u_3\}$, $\{v_1, v_2, v_3\}$ are subsets of B_n with maximal dimension, and if*

$$\dim \{u_i, u_j\} = \dim \{v_i, v_j\}, \qquad 1 \leqq i, j \leqq 3,$$

then $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are congruent modulo G_n.

Theorem 1 and the corollary imply that the number of incongruent sets of order 3 having maximal dimension in B_n is precisely the number of solutions of the following diophantine system:

$$2n = x + y + z, \qquad 1 \leqq x \leqq y \leqq z \leqq n.$$

THEOREM 2.

$$N_n^{(3)} = \sum_{k=2}^{n} \left\{ \left[\frac{k}{3} \right] + \sum_{r=0}^{[k/3]} \left[\frac{k-3r}{2} \right] \right\}.$$

PROOF. For each k, $2 \leqq k \leqq n$, it suffices to count the number of solutions, $N_k^{(3)} - N_{k-1}^{(3)}$, of

$$2k = x + y + z, \qquad 1 \leqq x \leqq y \leqq z \leqq k.$$

Suppose $z=k-r$, then $0 \leqq r \leqq [k/3]$ since $z=2k-x-y \geqq 2k-2z$, or $3z \geqq 2k$. Now $y \leqq z=k-r$ and $x=k+r-y \geqq k+r-(k-r)=2r$, but $x \leqq (k+r)/2$ so $2r \leqq x \leqq (k+r)/2$. For $r=0$, there are $[k/2]$ possible values for x; and if $r>0$, there are $[(k+r)/2-(2r-1)]$. Hence

$$\begin{aligned} N_k^{(3)} - N_{k-1}^{(3)} &= \left[\frac{k}{2}\right] + \sum_{r=1}^{[k/3]} \left[\frac{k-3r+2}{2}\right] \\ &= -1 + \sum_{r=0}^{[k/3]} \left[\frac{k-3r+2}{2}\right] \\ &= \left[\frac{k}{3}\right] + \sum_{r=0}^{[k/3]} \left[\frac{k-3r}{2}\right]. \end{aligned}$$

References

1. G. Pólya, J. Symbolic Logic, vol. 5 (1940) pp. 98–103.
2. D. Slepian, Canadian Journal of Mathematics vol. 5 (1953) pp. 185–193.

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