ON $u$-STABLE COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

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A commutative power-associative algebra $A$ of characteristic $>5$ with an idempotent $u$ may be written\(^1\) as the supplementary sum $A = A_u(1) + A_u(1/2) + A_u(0)$ where $A_u(\lambda)$ is the set of all $x_\lambda$ in $A$ with the property $x_\lambda u = \lambda x_\lambda$. The subspaces $A_u(1)$ and $A_u(0)$ are orthogonal subalgebras, $[A_u(1/2)]^2 \subseteq A_u(1) + A_u(0)$ and $A_u(\lambda)A_u(1/2) \subseteq A_u(1/2) + A_u(1 - \lambda)$ for $\lambda = 0, 1$. The algebra $A$ is called $u$-stable if $A_u(\lambda)A_u(1/2) \subseteq A_u(1/2)$ and is called stable if it is $u$-stable for every idempotent element $u$ of $A$.

A. A. Albert has shown in [3] that a simple commutative power-associative algebra $A$ of degree $>1$ over its center $F$ with characteristic prime to 30 is a Jordan algebra if and only if it is stable. Moreover, it is known that every simple algebra of degree $>2$ is a Jordan algebra. Thus there remains the problem of determining the nonstable simple algebras of degree two. There do exist simple algebras of characteristic $p > 5$ which are not Jordan algebras [3; 4]. Of course, these algebras are not stable, although they may be $u$-stable for some idempotent $u$. In this paper we shall obtain the following result.

THEOREM. Let $A$ be a $u$-stable simple commutative power-associative algebra of degree 2 over its center $F$ of characteristic zero. Then $A$ is a Jordan algebra.

We shall use all of the results of [3] so we shall adopt the notations of that paper. In particular, all the results of the section giving properties of $u$-stable algebras will be used. For convenience let us state a few of the required results here.

In a simple $u$-stable algebra $A$ there exists an element $w$ in $A_u(1/2)$ such that $w^2 = 1$. Then $A_u(1) = uB$, $A_u(0) = vB$, and $A_u(1/2) = wB + G$ where $B$ is the set of all elements $b$ of $C = A_u(1) + A_u(0)$ with the property $(wb)w = b$ and $G$ is the set of all quantities $g$ of $A_u(1/2)$ with the property $wg = 0$. Since $e = (1/2)(1 + w)$ and $f = 1 - e$ are orthogonal idempotents, we may decompose $A$ relative to $e$. It can be shown that $A_*(1) = eb$, $A_*(0) = fB$, and $A_*(1/2) = B(u - v) + G$. The set $B$ is a subalgebra of $C$ and the product of two elements in $G$ is in $B$. Also,

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\(^1\) The results of this paragraph are given in [1]. The numbers in brackets refer to the bibliography at the end of the paper.

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the following multiplicative relationships exist for any \( a, b \) in \( B, g \) in \( G \).

1. \( w(bu) = w(bv) = \frac{1}{2} wb, \)
2. \( (wa)b = w(ab), (wa)(wb) = ab, \)
3. \( g[b(u - v)] = wd, \)
4. \( gb = h - wc, \)
5. \( (wb)g + w(gb) = -d(u - v), \)
6. \( (wb)[a(u - v)] = k, \)

for \( h, k \) in \( G \), and \( c, d \) in \( B \). The quantity \( d \) in relation (5) is the \( d \) of (3).

The theorem can evidently be reduced to the case where \( F \) is algebraically closed. Then \( A_u(l) = uF + N_1 \) and \( A_u(0) = vF + N_0 \) where \( N_1 \) is the radical of \( A_u(\lambda) \) and \( N' = N_1 \oplus N_0 = N + N(u-v) \) is the radical of \( C \) where \( N \) is the radical of \( B \). Similarly, \( A_\varepsilon(1) = eF + M_1, A_\varepsilon(0) = fF + M_0, M_1 \) is the set of all elements \( ec \) where \( c \) is in \( N \) and we have the corresponding result for \( M_0 \).

The following important known lemma can now be stated.

**Lemma 1.** Let \( A \) be a commutative power-associative algebra of degree two over a field \( F \) of characteristic zero. Then \( A_\varepsilon(1/2)A_\varepsilon(1) \subseteq A_\varepsilon(1/2) + M_0 \) and \( A_\varepsilon(1/2)A_\varepsilon(0) \subseteq A_\varepsilon(1/2) + M_1 \). Note that the result of the lemma is not vacuous here since we are assuming \( u \)-stability only.

Consider the product \( (eB)G \) which was used to obtain (4) and (5). By Lemma 1, \( (eB)G \subseteq A_\varepsilon(1/2) + M_0 \) so that \( (b+wb)g = a(u-v) + h + c - wc \) for \( a, b \) in \( B, g, h \) in \( G \), and \( c \) in \( N \) the radical of \( B \). Then \( (wb)g = a(u-v) + c \) and it is shown in [3] that \( a = -d \) of relation (3). Also the quantity \( d \) in (3) and (5) is in \( N \). These results may be stated as follows.

**Lemma 2.** Let \( A \) be a \( u \)-stable commutative power-associative algebra over a field of characteristic zero. Then \( GB \subseteq G + wN, G[B(u-v)] \subseteq wN, w(GB) \subseteq N, (wB)G \subseteq N', \) and \( w(GB) + (wB)G \subseteq N(u-v) \).

It will also be necessary to have

**Lemma 3.** The product \( G\{ (wB)[B(u-v)] \} \subseteq N. \)

For proof substitute \( x = g, y = a, z = b(u-v) \) into the multilinear

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2 By Theorem 2 of [2].
3 See Theorem 6 of [5].
4 [2, Lemma 10].
identity obtained from the associativity of fourth powers.\footnote{The identity is stated in all of our references.} Relation (1) implies \( wz = w(az) = 0 \) and we have \( wg = 0 \) by definition of \( G \). Thus

\[
4(wa)(gz) = w[(ga)z + (gz)a + g(az)] + g[(wa)z] + a[(gz)w]
+ z[(wa)g + w(ga)].
\]

By (3) and (2), \((wa)(gz)\) is in \((wB)(wN) \subseteq N\). The quantity \( ga \) is in \( G+wN \) by (4); hence \( (ga)z \) is in \( G[B(u-v)]+(wN)[B(u-v)] \). Consequently, (3) and (6) imply \( w[(ga)z] \) in \( N \). Since \( (gz)a \) lies in \( \{G[B(u-v)]\}B \subseteq (wN)B \subseteq wN \), \( w[(gz)a] \) is in \( N \). Also \( w[g(az)] \) is in \( w \cdot G[B(u-v)] \subseteq w(wN) = N \). The product \( a[(gz)w] \) is in \( N \) and \( z[(wa)g + w(gz)] \) is contained in \( [B(u-v)] \cdot [N(u-v)] \subseteq N \). This completes the proof of Lemma 3.

The proofs of Lemmas 15 and 17 of [3] which state that \( [A_u(1/2) \cdot N'] \subseteq N'A_u(1/2) \) and \( [A_u(1/2)N']A_u(1/2) \subseteq N' \) follow without change. We also have without change that \( N' + A_u(1/2)N' \) is an ideal of \( A \). Since \( A \) is simple, this ideal must be zero because it does not contain the identity element. Thus \( A = uF + vF + A_u(1/2) \), which is a Jordan algebra. A Jordan algebra is stable so we have as a corollary that a simple commutative power-associative algebra of degree 2 and characteristic 0 is stable if and only if it is \( u \)-stable.

\textbf{Bibliography}


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