

schitz condition of any positive order [3, p. 136], and more general conditions on  $f^{(4)}(x)$  are known [3, p. 143].

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## ON $n$ -POINT EXPANSIONS OF ENTIRE FUNCTIONS

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In a recent paper [5], A. J. MacIntyre has studied the representation of entire functions of exponential type in certain  $n$ -point expansions associated with the names Lidstone, Whittaker, Gontcharoff, and Poritsky. In this note, I shall show that the kernel expansion method which I have previously used to discuss general expansion theorems is also applicable to the cases which he has studied.

The general class of expansion formulae that concern us arise from the problem of expressing a function  $f$  in terms of the values of a sequence of its derivatives, computed at a finite set of points. Let  $\{L_n\}$  be a sequence of functionals defined on the class  $K$  of functions of exponential type by

$$(1) \quad L_{pk+j}(f) = f^{(pk+rj)}(\beta_j) \quad k = 0, 1, \dots; j = 0, 1, \dots, p-1.$$

This sequence is specified by the set of integers  $\{r_0, r_1, \dots, r_{p-1}\}$  and the set of complex numbers  $\{\beta_0, \beta_1, \dots, \beta_{p-1}\}$ . The Lidstone set [10] arise from the choice  $p=2$ ,  $r_0=r_1=0$ ,  $\beta_0=0$ ,  $\beta_1=1$ . The generalization introduced by Poritsky [7] corresponds to  $r_j=0$  for  $j=0, 1, \dots, p-1$  with arbitrary  $\beta_j$ . The Whittaker two-point set [10] arises when  $p=2$ ,  $r_0=0$ ,  $r_1=1$ ,  $\beta_0=1$ ,  $\beta_1=0$ , and the periodic Gontcharoff set corresponds to  $r_j=j$  with arbitrary  $\beta_j$  [4].

The general problem is to discover to what extent knowledge of the values  $L_n(f)$  for  $n=0, 1, \dots$  determine  $f$ . If  $f$  is of exponential

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type at most  $A$ , then

$$(2) \quad f(z) = \frac{1}{2\pi i} \int_{\Gamma} e^{zw} F(w) dw$$

where  $F$  is the Laplace transform of  $f$  and is regular at least for  $|w| > A$ . The path  $\Gamma$  encloses the complete set of singularities of  $F$ . Using this, the functionals  $L_n$  have the representation

$$(3) \quad L_n(f) = \frac{1}{2\pi i} \int_{\Gamma} u_n(w) F(w) dw$$

where

$$(4) \quad u_{pk+j}(w) = (w^p)^k w^{rj} e^{\beta j w}.$$

To apply the method of kernel expansion in the present case, we seek an expansion of  $e^{zw}$  in the form  $e^{zw} = \sum_0^{\infty} p_n(z) u_n(w)$  which is uniformly convergent (or summable) for  $w$  on  $\Gamma$ , and for all  $z$  in compact sets. Combining this with (2), we have  $f(z) = \sum_0^{\infty} L_n(f) p_n(z)$  with  $L_n(f)$  given by (3).

Since the central point of the present note is to discuss methods rather than detailed results, we treat the representative Lidstone case alone [1; 8; 9; 10; 11]. We therefore seek to write  $e^{zw}$  in the form  $\sum_0^{\infty} p_n(z) w^{2n} + \sum_0^{\infty} q_n(z) w^{2n} e^w$ . It is easily seen that

$$(5) \quad e^{zw} = \frac{\sinh(1-z)w}{\sinh w} + e^w \frac{\sinh zw}{\sinh w}.$$

The function  $\sinh zw / \sinh w$  is regular for  $|w| < \pi$  and is even in  $w$  so that it has the power series development  $\sum_0^{\infty} \Lambda_n(z) w^{2n}$  convergent for  $|w| < \pi$ , where  $\Lambda_n$  is a polynomial. Using this, we have  $e^{zw} = \sum_0^{\infty} \Lambda_n(1-z) w^{2n} + \sum_0^{\infty} \Lambda_n(z) w^{2n} e^w$ , where both series converge for  $|w| < \pi$ . Proceeding as above (or in [2] or [8]) we conclude that any function  $f$  of exponential type  $A < \pi$  has the convergent expansion  $f(z) = \sum_0^{\infty} \Lambda_n(1-z) f^{(2n)}(0) + \sum_0^{\infty} \Lambda_n(z) f^{(2n)}(1)$  which is the desired Lidstone series for  $f$ . As indicated in [2] or in [5], summability may also be discussed. From this Lidstone expansion, we see that if  $f$  is of type less than  $\pi$  and if  $f^{(2n)}(0) = f^{(2n)}(1) = 0$  for  $n = 0, 1, 2, \dots$ , then  $f(z) \equiv 0$ . Schoenberg [9] using a theorem of Carmichael on difference equations [3] showed that the only functions of exponential type which obey the conditions  $f^{(2n)}(0) = f^{(2n)}(1) = 0$  are the functions  $\sin \pi z, \sin 2\pi z, \dots$  and their linear combinations. By combining the discussion given above with a device used by W. T. Martin [6] these results may be obtained easily. We observe that the limitation

$|w| < \pi$  arises from the fact that  $\sinh w$  which occurs in (5) has simple zeros at  $\pm \pi i$ . The residue of  $\sinh zw / \sinh w$  at  $\pm \pi i$  is  $\mp i \sin \pi z$  so that

$$\frac{\sinh zw}{\sinh w} = 2\pi \frac{\sin \pi z}{w^2 + \pi^2} + G_1(z, w)$$

where  $G_1(z, w)$  is regular for  $|w| < 2\pi$  and even in  $w$ . More generally,

$$\frac{\sinh zw}{\sinh w} = \pi \sum_1^N \frac{(-1)^{n+1} n \sin(n\pi z)}{w^2 + n^2\pi^2} + G_N(z, w)$$

where  $G_N(z, w) = \sum_0^\infty g_n(z)w^{2n}$ , convergent for  $|w| < (N+1)\pi$ . The function  $g_n(z)$  is the sum of the polynomial  $\Lambda_n(z)$  and a finite trigonometric sum. This in turn gives

$$\begin{aligned} e^{zw} &= \sum_0^\infty g_n(z)w^{2n}e^w + \sum_0^\infty g_n(1-z)w^{2n} \\ &\quad + \pi \sum_1^N \frac{n \sin(n\pi z)}{w^2 + n^2\pi^2} [1 + (-1)^{n+1}e^w] \end{aligned}$$

convergent for  $|w| < (N+1)\pi$  and we obtain in the same way as before the following result.

**THEOREM.** *If  $f$  is of exponential type  $A < (N+1)\pi$  then*

$$f(z) = \sum_0^\infty g_n(1-z)f^{(2n)}(0) + \sum_0^\infty g_n(z)f^{(2n)}(1) + \sum_1^N C_n \sin(n\pi z)$$

where

$$C_n = \frac{n}{2i} \int_\Gamma \frac{1 + (-1)^{n+1}e^w}{w^2 + n^2\pi^2} F(w)dw$$

and  $\Gamma$  may be taken as the circle  $|w| = \rho$  with  $A < \rho < (N+1)\pi$ .

**COROLLARY.** *If  $f$  is of exponential type  $A < (N+1)\pi$  and  $f^{(2n)}(0) = f^{(2n)}(1) = 0$  for  $n = 0, 1, 2, \dots$  then  $f(z) = \sum_1^N C_n \sin(n\pi z)$ .*

It is clear that the same approach may be used to discuss the other cases treated by MacIntyre [5] as well as the more general case studied incompletely by Schmidli [8] and that it has many advantages over that used by the former.

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