# INEQUALITIES INVOLVING CYLINDRICAL FUNCTIONS

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Watson [1] derives the following equations for Bessel functions, valid for real positive $\nu$ and $0<x\leqq 1$.

$$J_\nu(\nu x) = \frac{1}{\pi}\int_0^\pi \exp\{-\nu F(\theta, x)\}\, d\theta, \tag{1}$$

$$J_\nu'(\nu x) = \frac{1}{\pi}\int_0^\pi \frac{\theta - x^2 \sin\theta\cos\theta}{x(\theta^2 - x^2\sin^2\theta)^{1/2}} \exp\{-\nu F(\theta, x)\}\, d\theta, \tag{2}$$

where

$$F(\theta, x) = \log\frac{\theta + (\theta^2 - x^2\sin^2\theta)^{1/2}}{x \sin\theta} - \cot\theta\left[(\theta^2 - x^2\sin^2\theta)^{1/2}\right],$$

$$0 \leqq \theta \leqq \pi.$$

Using these relations Watson derives

$$J_\nu(\nu x) \leqq \frac{\exp\{-\nu F(0, x)\}}{(1 - x^2)^{1/4}(2\pi\nu)^{1/2}}, \tag{3}$$

$$J_\nu'(\nu x) \leqq (1 + x^2)^{1/4}\frac{\exp\{-\nu F(0, x)\}}{x(2\pi\nu)^{1/2}}, \tag{4}$$

$$J_\nu(\nu) < \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi\nu^{1/3}}. \tag{5}$$

One of the authors has shown that (3) may be replaced by an inequality which remains finite at $x=1$ [2]. Equations (3) and (4) have been extended to the case in which the argument is nearly equal to and larger than the order [3].

In this paper inequalities are derived relating $J_\nu(\nu x)$ and $J_\nu'(\nu x)$ to the Bessel functions of equal order and argument, $J_\nu(\nu)$ and $J_\nu'(\nu)$.

It is shown that

$$J_\nu(\nu x) \leqq \exp\{-\nu F(0, x)\} J_\nu(\nu), \tag{6}$$

$$J_\nu'(\nu x) \leqq \frac{(1 - x^2)^{1/2}}{x} J_\nu(\nu x) + x J_\nu'(\nu)\exp\{-\nu F(0, x)\}, \tag{7}$$

$$J_\nu'(\nu) < \frac{3^{1/6}\Gamma(2/3)}{2^{1/3}\pi\nu^{2/3}} \approx \frac{0.41085}{\nu^{2/3}}. \tag{8}$$

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The right-hand members of (6) and (7) not only remain finite as $x \to 1$ but also approach $J_\nu(\nu)$ and $J'_\nu(\nu)$ respectively. Equation (6) is an improvement of (3) when

$$J_\nu(\nu)(1 - x^2)^{1/4} \leqq 1/(2\pi\nu)^{1/2} \tag{9}$$

while (7) is an improvement of (4) when

$$J'_\nu(\nu) < \frac{(1 + x^2)^{1/4} - (1 - x^2)^{1/4}}{x^2(2\pi\nu)^{1/2}}. \tag{10}$$

It can be observed

$$\frac{(1 + x^2)^{1/4} - (1 - x^2)^{1/4}}{x^2(2\pi\nu)^{1/2}} > \frac{0.199}{\nu^{1/2}} .$$

Now comparing this with (8), (7) is an improvement of (4) for all $x$ whenever $\nu > 78.5$ approximately.

**Derivation.**

PROOF OF (6).

$$\frac{\partial}{\partial x} F(\theta, x) = - \frac{\theta - x^2 \sin\theta \cos\theta}{x(\theta^2 - x^2 \sin^2\theta)^{1/2}},$$

$$\frac{\partial}{\partial x} F(0, x) = - \frac{(1 - x^2)^{1/2}}{x} \qquad [1, p. 253].$$

Now it is easy to show that

$$\frac{\theta - x^2 \sin\theta \cos\theta}{x(\theta^2 - x^2 \sin^2\theta)^{1/2}} \geqq \frac{(1 - x^2)^{1/2}}{x},$$

that is

$$\frac{\partial F(0, x)}{\partial x} \geqq \frac{\partial F(\theta, x)}{\partial x} .$$

Integrating from $x$ to 1 and using the fact that $F(0, 1) = 0$

$$F(\theta, x) \geqq F(\theta, 1) + F(0, x). \tag{11}$$

Therefore

$$J_\nu(\nu x) \leqq \frac{1}{\pi} \int_0^\pi \exp\{-\nu[F(\theta, 1) + F(0, x)]\} d\theta \tag{12}$$

whence follows (6).

COROLLARY. *Using* (5) *we obtain*

$$(13) \qquad J_\nu(\nu x) < \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi\nu^{1/3}} \exp\left\{-\nu F(0, x)\right\}.$$

PROOF OF (7). Let

$$L(\theta, x) = \frac{\theta - x^2 \sin\theta\cos\theta}{(\theta^2 - x^2\sin^2\theta)^{1/2}}.$$

Then

$$(14) \qquad L(\theta, x) = \frac{\theta(1 - x^2)}{(\theta^2 - x^2\sin^2\theta)^{1/2}} + \frac{x^2(\theta - \sin\theta\cos\theta)}{(\theta^2 - x^2\sin^2\theta)^{1/2}}.$$

Using

$$(15) \qquad (\theta^2 - x^2\sin^2\theta)^{1/2} \geqq \theta(1 - x^2)^{1/2},$$

and

$$(16) \qquad (\theta^2 - x^2\sin^2\theta)^{1/2} \geqq (\theta^2 - \sin^2\theta)^{1/2} \qquad \text{in (14)},$$

$$L(\theta, x) \leqq (1 - x^2)^{1/2} + x^2 L(\theta, 1).$$

Hence

$$(17) \qquad \begin{aligned} J_\nu'(\nu x) &\leqq \frac{1}{\pi x}\int_0^\pi \left\{(1 - x^2)^{1/2} + x^2 L(\theta, 1)\right\} \exp\left\{-\nu F(\theta, x)\right\} d\theta \\ &\leqq \frac{(1 - x^2)^{1/2}}{x} J_\nu(\nu x) + \frac{x}{\pi}\int_0^\pi L(\theta, 1)\exp\left\{-\nu F(\theta, x)\right\} d\theta \\ &\leqq \frac{(1 - x^2)^{1/2}}{x} J_\nu(\nu x) \\ &\quad + \frac{x}{\pi}\int_0^\pi L(\theta, 1)\exp\left\{-\nu[F(0, x) + F(\theta, 1)]\right\} d\theta \\ &\leqq \frac{(1 - x^2)^{1/2}}{x} J_\nu(\nu x) + xJ_\nu'(\nu)\exp\left\{-\nu F(0, x)\right\}, \end{aligned}$$

and (7) is proved.

COROLLARY. *Using* (8) *and* (5)

$$(18) \qquad \begin{aligned} J_\nu'(\nu x) \leqq{} & \frac{(1 - x^2)^{1/2}}{x} \frac{\Gamma(1/3)}{2^{2/3}3^{1/6}\pi\nu^{1/3}} \exp\left\{-\nu F(0, x)\right\} \\ & + \frac{3^{1/6}\Gamma(2/3)}{2^{1/3}\pi\nu^{2/3}} x \exp\left\{-\nu F(0, x)\right\}. \end{aligned}$$

PROOF OF (8). From the inequality

$$(2\theta^2 - \sin^2\theta)\cos\theta \leqq \theta\sin\theta \qquad (0 \leqq \theta \leqq \pi) \quad [1, \text{p. } 256]$$

it follows that $L(\theta, 1)/\theta$ is monotone decreasing $(0 \leqq \theta \leqq \pi)$. Hence

$$L(\theta, 1)/\theta \leqq \lim_{\theta\to 0} L(\theta, 1)/\theta = 2/3^{1/2}.$$

Now

$$J_\nu'(\nu) = \frac{1}{\pi}\int_0^\pi L(\theta, 1)\exp\{-\nu F(\theta, 1)\}\,d\theta,$$

and since

$$F(\theta, 1) \geqq 4\theta^3/9\cdot 3^{1/2} \qquad [1, \text{p. } 259],$$

$$J_\nu'(\nu) < \frac{1}{\pi}\int_0^\infty \frac{2}{3^{1/2}}\exp\left\{-\nu\frac{4\theta^3}{9\cdot 3^{1/2}}\right\}d\theta < \frac{3^{1/6}\Gamma(2/3)}{2^{1/3}\pi\nu^{2/3}}\cdot$$

The right-hand side of the last inequality is the first term of Cauchy's asymptotic series for $J_\nu'(\nu)$.

**Further remarks.** The recursion formulas can now be used as in [3] to produce inequalities for the case in which the argument of the Bessel function is greater than and nearly equal to the order.

This results in

$$(19) \qquad J_{\nu-1}(\nu x) \leqq \left[\frac{1+(1-x^2)^{1/2}}{x}J_\nu(\nu) + xJ_\nu'(\nu)\right]\exp\{-\nu F(0, x)\}$$

valid for $1-1/\nu < x \leqq 1$.

Once again this expression becomes an equality when $x=1$.

Similar inequalities may be derived for the Neumann function. For example, if $\gamma_\nu$ is defined by

$$J_\nu(\nu) = -N_\nu(\nu)\tan\gamma_\nu,$$

it is shown in [1, p. 515] that $\gamma_\nu$ is an increasing function of $\nu$ for all positive $\nu$ and that $\lim_{\nu\to\infty}\gamma_\nu = \pi/6$. Furthermore for so small a value as $\nu=12$, $\gamma=29°\ 58\text{-}1/2'$. Hence for $\nu \geqq 12$

$$J_\nu(\nu) \geqq -.5766\,N_\nu(\nu)$$

or

$$|N_\nu(\nu)| \leqq 1.73\,J_\nu(\nu).$$

It is also shown in [1, p. 487] that in the interval $(0, j_\nu')$, where $j_\nu'$ denotes the first zero of the derivative of $J_\nu(x)$, $N_\nu(x)$ is negative and increasing. Hence for $j_\nu'/\nu \geqq x \geqq 1$, a simple bound for $N_\nu(\nu x)$ is

$$(20) \qquad |N_\nu(\nu x)| \leqq |N_\nu(\nu)| \leqq 1.73|J_\nu(\nu)|.$$

## References


1. G. N. Watson, *A treatise on the theory of Bessel functions*, 2d ed., Cambridge University Press, 1944.

2. K. M. Siegel, *An inequality involving Bessel functions of argument nearly equal to their order*, Proc. Amer. Math. Soc., vol. 4 (1953) pp. 858–859.

3. K. M. Siegel and F. B. Sleator, *Inequalities involving cylindrical functions of nearly equal argument and order*, Proc. Amer. Math. Soc. vol. 5 (1954) pp. 337–344.


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# NEWTON'S METHOD IN BANACH SPACES[1]

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In this note we show that if $f$ is a mapping between Banach spaces which is in class $C'$ in the sense of Hildebrandt and Graves [4],[2] then the equation $f(x)=0$ may be solved by an iterative process:

$$x_{n+1} = x_n - [f'(z_n)]^{-1}f(x_n), \qquad n = 0, 1, 2, \cdots,$$

provided that the initial guess $x_0$ and the arbitrarily selected points $z_n$ are sufficiently close to the solution desired. Here the derivative is taken in the sense of Fréchet. If we let $z_n = x_n$, $n=0, 1, 2, \cdots$, we obtain the usual Newton process; if $z_n = x_0$, $n=0, 1, 2, \cdots$, we obtain what is sometimes called the modified Newton process. Naturally, in any application, the computer would determine the $z_n$ so as to minimize effort.

This result is closely related to recent theorems of Kantorovič [5; 6; 7] and Mysovskih [8; 9], although these authors assumed the existence and boundedness of the second Fréchet derivative of $f$. In turn for this assumption, they were able to establish more rapid convergence. Under the assumption of analyticity, Stein [10] eliminated explicit mention of the second derivative, which is desirable

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[1] This paper was written while the author was on Contract nonr 609(04) with the Office of Naval Research.

[2] Numbers in brackets refer to the list of references at the end.