BOUNDED SETS IN (*F*)-SPACES

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1. In the theory of (F)-spaces,[1] the following two questions arise:

(A) Let E be a metrizable locally convex space; is it true that if *every bounded set* in E is separable, the space E is separable?

(B) Let E be a metrizable locally convex space, $\overline{E}$ its completion; is it true that every bounded set in $\overline{E}$ is contained in the *closure* of a bounded set of E?

The purpose of this note is to prove that, *if we assume the continuum hypothesis*, the answer to both questions is *negative*.

2. We begin by proving a lemma which is substantially well known (see e.g. [**2**]; the continuum hypothesis is used at this place).

LEMMA. *Let us well-order the set $\boldsymbol{N}^{\boldsymbol{N}}$ of all sequences of positive integers in a transfinite sequence (f_λ) of type ω_1; then there is a subset A of ordinals, having the power of the continuum, such that if α, β are in A and $\alpha<\beta$, $f_\alpha=o(f_\beta)$, and that, for every ordinal $\lambda<\omega_1$, there is an $\alpha>\lambda$ in A such that $f_\lambda=o(f_\alpha)$. We can moreover suppose that f_α is a strictly increasing sequence for every $\alpha\in A$.*

Suppose we have defined all elements of A which are $<\alpha$; the ordinals $<\alpha$ constituting a denumerable set, there is, by Du Bois-Reymond's theorem, a strictly increasing sequence f_λ with $f_\mu=o(f_\lambda)$ for every $\mu<\alpha$; ordinals λ having that property are obviously $\geqq\alpha$, and we take as the first element of A which is $\geqq\alpha$, the smallest of these ordinals λ. It is clear that A has the power of the continuum.

As a corollary, it follows that if B is a subset of A such that there exists an ordinal $\alpha\in A$ such that $f_\beta=O(f_\alpha)$ for every $\beta\in B$, then B is *denumerable*, for the indices of B are necessarily $<\alpha$.

3. In the product space $\boldsymbol{R}^{\boldsymbol{N}}$ (space of all sequences of real numbers, with the topology of pointwise convergence), let us consider the set S of all elements of the form $g_n(f_\alpha+m)$, where $\alpha\in A$, m is an arbitrary integer (positive or negative), and g_n (for every integer $n\geqq 0$) is the sequence $(\xi_{nk})_{k\geqq 0}$ where $\xi_{nk}=0$ for $k<n$, $\xi_{nk}=1$ for $k\geqq n$. In other words, $g_n(f_\alpha+m)$ is the sequence (ξ_k) such that $\xi_k=0$ for $k<n$, $\xi_k=f_\alpha(k)+m$ for $k\geqq n$. On the other hand, let F be the Banach space of all real *bounded* functions $(x_\alpha)_{\alpha\in A}$, with the usual norm $\sup_{\alpha\in A}|x_\alpha|$. For each $\alpha\in A$, let e_α be the element of F, whose coordinates are all 0, ex-

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[1] We use the terminology of [**1**] and [**3**].

cept that of index α, which is equal to 1; we observe that $\|e_\alpha\|=1$ and that the distance of e_α to the closed subspace of F generated by the e_β of index $\beta\neq\alpha$ is 1.

In the product space $F\times\boldsymbol{R}^N$, consider now the set of all elements $(e_\alpha,\ g_n(f_\alpha+m))$, where α ranges over A, n over all integers $\geqq 0$, m over all positive or negative integers; and let E be the subspace of $F\times\boldsymbol{R}^N$ generated by all *finite* linear combinations of these elements. It is clear that E is locally convex and metrizable; on the other hand, the projection of E on F is not separable, hence *E is not separable.*

Let now H be a bounded set in E; its projection L on $\boldsymbol{R}^N$ is then bounded, which means that there exists an ordinal $\mu<\omega_1$ such that all elements of L are $O(f_\mu)$. Now it is clear that every element $z=(x,\ y)$ of E can be written in one and only one way

$$z=\sum_i \lambda_i(e_{\alpha_i},\ f_{\alpha_i})+(0,\ u)$$

where u belongs to the set U of all *ultimately constant* sequences (i.e., constant after a certain term), the α_i are distinct ordinals of A, finite in number (eventually equal to 0), and the λ_i are $\neq 0$. If $z\notin U$, we have $y\sim\lambda_k f_{\alpha_k}$, where α_k is the largest of the α_i; hence in all elements $z\in H$ which are not in U, the α_i are bounded by a common $\beta\in A$ (from the lemma in §2), and therefore constitute a *denumerable* set. As U is obviously a separable subset of $\boldsymbol{R}^N$, we have proved that *each bounded subset H of E is separable*, thus getting a negative answer to problem (A).

4. Let now $\overline{E}$ be the completion of E, which we can identify with the closure of E in the product space $F\times\boldsymbol{R}^N$. Let B be the bounded subset of $\overline{E}$ consisting of all elements $(x,\ y)$ of that space such that $\|x\|\leqq 2$ and $|y(n)|\leqq 1$ for every n. We are going to prove that *B is not separable*, thus answering problem (B) negatively.

Let α be an arbitrary element of A; as A is nondenumerable, there is an infinite sequence (α_k) of elements of A such that $\alpha_0=\alpha$, $\alpha_k<\alpha_{k+1}$ hence $f_{\alpha_k}=o\ (f_{\alpha_{k+1}})$. We are going to determine a convergent sequence (h_k) in the space $\boldsymbol{R}^N$ in the following way. Let the integer m_0 be determined such that $|f_{\alpha_0}(0)+m_0|\leqq 1$, and take $h_0=g_0(f_{\alpha_0}+m_0)$. Let n_1 be the first integer for which $|h_0(n)|>1$; we determine the integer m_1 such that $f_{\alpha_1}(n_1)+m_1\geqq 2|h_0(n_1)|$, and finally, we take $h_1=h_0+\lambda_1 g_{n_1}\cdot(f_{\alpha_1}+m_1)$, λ_1 being a real number determined by the equation $h_1(n_1)=0$; it is clear that $|\lambda_1|\leqq 1/2$. Let n_2 be the first integer for which $|h_1(n)|>1$; we take m_2 such that $f_{\alpha_2}(n_2)+m_2\geqq 2^2|h_1(n_2)|$, and then we form $h_2=h_1+\lambda_2 g_{n_2}(f_{\alpha_2}+m_2)$, λ_2 being determined by the equation $h_2(n_2)=0$; we have $|\lambda_2|\leqq 1/2^2$. The induction proceeds in an obvious fashion, and, due to the choice of the α_k, is never stopped. Moreover, $h_k(n)$ has, for each n, a fixed value from a certain k on, hence the sequence (h_k) converges in $\boldsymbol{R}^N$ to an element y_α such that $|y_\alpha(n)| \leqq 1$ for each n. But h_k is the projection of the element

$$(e_{\alpha_0} + \lambda_1 e_{\alpha_1} + \cdots + \lambda_k e_{\alpha_k}, h_k)$$

of E, and the sequence of these elements obviously converges to an element (t_α, y_α) of $\overline{E}$ which belongs to B. To each $\alpha \in A$ we have thus attached an element $z_\alpha = (t_\alpha, y_\alpha)$ of B; moreover the construction is such that, if α, β are two indices in A such that $\alpha < \beta$, we have $\|t_\alpha - t_\beta\| \geqq 1$. The projection of B on F is thus a nonseparable space, and so therefore is B itself.

5. It would be interesting to give negative answers to problems (A) and (B) without using the continuum hypothesis. On the other hand, if in the statement of problem (A), we require that E be *complete* (in other words, an (F)-space), we obtain a modified problem (A′) which (even with the continuum hypothesis) remains still unsolved.

Bibliography

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