RIGHT ANNIHILATOR ALGEBRAS

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1. **Introduction.** The purpose of this note is to extend to arbitrary Banach spaces the characterization given by Bonsall and Goldie [1; 2] of the Banach algebra $F(X)$ of all approximately finite-valued operators on a reflexive Banach space X. This extension was suggested by one of the results of Bonsall and Goldie [2, Theorem 15]. The requirements on a topologically simple Banach algebra A in order that it be isometric and isomorphic to some $F(X)$ are that every left ideal which is not dense in A has a nonzero right annihilator and a certain restriction on the spectral radius of some nonzero multiple of every element of A. (See Condition (3).) Our arguments are merely refinements and simplifications of those of Bonsall and Goldie. Although we have made some effort to make our one-handed presentation fairly self-contained, the reader is assumed to have a knowledge of the arguments employed by Bonsall and Goldie.

We devote §1 to the reduction of our problem (in a slightly more general setting) to the topologically simple case. In §2 we offer refinements of some of the arguments of Bonsall and Goldie which lead to the desired characterization.

As to notation, we let $E_r(E_l)$ denote, for a subset of a ring A, all those x in A for which $Ex=0$ $(xE=0)$. If F is also a subset of A, EF denotes the totality of products xy with x in E and y in F, while $E\cdot F$ denotes the additive subgroup of A generated by EF. When A is a topological ring, E^- denotes the closure of E. We call E *modular* in case $A(1-f)\subset E$ for some f in A. None of this notation or terminology is new.

2. **Semi-simple right annihilator rings.** We shall call a topological ring A a *right annihilator ring* in case

(1) If I is a left ideal of A and $I^-\neq A$, then $I_r\neq 0$,

(2) Each modular maximal left ideal of A is closed.

Observe that it is a consequence of (1) and (2) that $I_r\neq 0$ for each proper modular left ideal I of A because we may imbed I in a modular maximal left ideal and apply (2) and (1). We shall assume throughout this section that A is semi-simple.

If a in A is not left quasi-regular (l.q.r.), then $A(1-a)$ is a proper modular left ideal and hence $(A(1-a))_r=[1-a]_r\neq 0$, or $ab=b\neq 0$ for some b in A. Now let M be a modular maximal left ideal of A so

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that $M_r \neq 0$. Not every element in M_r is l.q.r., because if every element of M_rA is l.q.r. so is every element of AM_r [4, p. 154], M_r is contained in the radical of A, $M_r=0$, a contradiction. Let e in M_r not be l.q.r. so that $eb=b\neq 0$ for some b in A. Then e is not in $M+A(1-e)$, for $e=m+x-xe$ with m in M and x in A yields $eb=0$ $=b$, a contradiction. Since M is maximal, $A(1-e)\subset M$, $A(1-e)e=0$, $e=e^2$ is idempotent in M_r. Then $M\subset [a]_l=A(1-e)$ gives $M=A(1-e)$, $M_r=eA$. Since M is maximal, Ae is minimal and so is eA [5, p. 13]. Because $(M_r)_l\supset M$ and is maximal [5, p. 13], we obtain $(M_r)_l=M$. Thus one of the dual rules of Kaplansky [4] holds for modular maximal left ideals of A. It is not known whether this rule holds for all closed left ideals of A (cf. [2]).

If J is a left ideal of A and $J^- \neq A$, then J_r contains a minimal right ideal. For, J_r contains an element a which is not l.q.r. and we have $ab=b\neq 0$ for some b in A. Then, as in the previous paragraph, a is not in $J+A(1-a)$ and there is a modular maximal left ideal M such that $M\supset J+A(1-a)$. Then M_r is a minimal right ideal and $M_r\subset J_r$. We apply this to the join J of all the minimal left ideals of A and obtain $J^-=A$, for otherwise J_r would contain a minimal right ideal whose idempotent generator would have to be in J, a contradiction. If I is a minimal left ideal of A, then the closure of the two-sided ideal of A generated by I, $\{I\}$, is a minimal closed two-sided ideal of A, and every minimal closed two-sided ideal B of A has this form. For, if $B\cap I\neq 0$, then $I\subset B$, $\{I\}=B$, while $B\cap I=0$ for every minimal left ideal I of A yields $bJ=0$, $bA=0$, $b=0$ for every b in B, a contradiction. Thus A is the closure of the join of its minimal closed two-sided ideals, and this join is clearly direct.

Now let B be one of the minimal closed two-sided ideals of A and let I be a left ideal of B such that $K=I^-\neq B$. Then K is a left ideal of A, and so is $L=K+B_l$. (This proves that B is topologically simple.) If $L^-=A$, and b, $b'\in B$, then every neighborhood of b contains $k+c$ with k in K and c in B_l. If W is a neighborhood of $b'b$, then $b'Z\subset W$ for some neighborhood Z of b and hence $b'k\in W$, $b'b\in K^-=K$. But then $(B\cdot B)^-\subset K$, $B\subset K$, a contradiction. Thus $L^-\neq A$ and L_r contains a minimal right ideal eA. If e is not in B, then $Be=0$, $e\in B_r=B_l$ (because $B_rB\subset B\cap B_r=0$, $B_r\subset B_l$, and dually), $e\in L$, $e=0$, a contradiction. Hence $e\in B$ and $Le=Ke=0$, $K_r\cap B\neq 0$. Finally, let N be a modular maximal left ideal of B so that $B(1-f)\subset N$ for some f in B and hence f not in N. Then f is not in $N+A(1-f)$ since $f=n+a-af$ with a in A and n in N yields $f^2=fn+fa-faf$ in N, which with $f-f^2$ in N gives f in N, a contradiction. Thus there is a modular maximal left ideal M of A such that $M\supset N+A(1-f)$ and hence f is not in

$M=M^-$. Suppose that $f\in N^-$, then $f^2\in AN^-\subset AM^-=AM\subset M$, $f-f^2\in M$, $f\in M$, a contradiction. Since N is maximal in B and B is closed, it follows that N is closed.

When we recall that every ideal B of a semisimple ring A is also semisimple (Direct proof: If $b\in B$ and bc is r.q.r. for every c in B, then $baba$ is r.q.r. for every a in A and thus ba is r.q.r. for every a in A, $b=0$), we may summarize the results of this section in a theorem.

THEOREM 1. *If A is a semi-simple topological ring which satisfies conditions* (1) *and* (2), *then A is the closure of the direct join of its minimal closed two-sided ideals, each of which is a semi-simple and topologically simple ring which satisfies* (1) *and* (2).

3. **Topologically simple right annihilator algebras.** We now impose, in addition to the assumptions of §1, the requirements that A is a Banach algebra and that A is topologically simple. (Of course, condition (2) is now redundant.) If eA is a minimal right ideal of A, with e idempotent, and $xR_a=xa$ for x in eA, then $a\rightarrow R_a$ is a continuous isomorphism of A onto the ring R_A, provided we employ the uniform topology for operators in the Banach space $X=eA$. Bonsall and Goldie show, without using the dual of condition (1), that R_A includes all finite-valued operators on X [2, Theorem 10]. It does not seem to be known whether R_A is necessarily closed. However, $\|a\|_1=\|R_a\|$ yields a second norm for A such that $\|a\|_1\leqq\|a\|$ for every a in A. In order to secure a faithful representation of A, we impose the following generalization of the requirement of Bonsall [1, Equation (2.1)].

(3) If $a\in A$ with $\|a\|=1$ and $0<\epsilon<1$, then for some $a^\#$ in A with $\|a^\#\|=1$, we have $\|(aa^\#)^n\|\geqq(1-\epsilon)^n$ for every positive integer n.

That (3) holds in $F(X)$ may be seen by a trivial modification of the proof of Theorem 1 of [1]. It is also clear that if (3) holds for a family of Banach algebras, it also holds in the completion of their direct join ($=B(\infty)$ sum). When A is not topologically simple, the condition (3) for A implies that each minimal closed two-sided ideal B of A satisfies condition (3). For consider $b\in B$ with $\|b\|=1$ and $0<\epsilon<1$. With $\delta=1-(1-\epsilon)^{1/2}$, we obtain $a^\#$ in A with $\|a^\#\|=1$ and $\|(ba^\#ba^\#)^n\|\geqq(1-\epsilon)^n$ for every n. We note that $1\geqq\|a^\#ba^\#\|\geqq\|ba^\#ba^\#\|\geqq1-\epsilon>0$, and we may set $b^\#=a^\#ba^\#/\|a^\#ba^\#\|$ in B to find that $\|(bb^\#)^n\|\geqq(1-\epsilon)^n$, as desired. (One should observe that an analogous argument applies in the $B^\#$ algebras of Bonsall.)

Let us return to the topologically simple case and use condition (3). Following Bonsall, we let A_1 be the completion of A relative to $\|a\|_1$ and prove that the spectral radius, $r(a)$ of a in A is equal to the spectral radius, $r_1(a)$, of a in A_1. It is sufficient to observe that it is impossible that an element b of A be q.r. in A_1 and not q.r. in A. For, if b is l.q.r. in A, it is also r.q.r. in A by the uniqueness of the quasi-inverse in A_1. But when b is not l.q.r., $bc = c \neq 0$ for some c in A and b cannot be q.r. in A_1. Next we prove that $\|a\| = \|a\|_1$ for every a in A. For suppose that $\|a\|_1 = 1 - \delta$ with $\|a\| = 1$ and $0 < \delta < 1$, then with $\epsilon = \delta/2$, we obtain $a^\#$ in A with $\|a^\#\| = 1$ and $\|(aa^\#)^n\| \geqq (1-\epsilon)^n$. Then we have $(1-\delta)\|a^\#\|_1 \geqq \|aa^\#\|_1 \geqq r_1(aa^\#) = r(aa^\#) \geqq \lim \|(aa^\#)^n\|^{1/n} \geqq 1 - \epsilon$. Since $1 - \epsilon > 1 - \delta$, this yields $\|a^\#\|_1 > \|a^\#\| = 1$, a contradiction. Thus $\|a\|_1 = \|a\|$ for every a in A and the mapping $a \rightarrow R_a$ is an isometric isomorphism. We note that (3) implies the semi-simplicity of A (cf. [1, Theorem 4]) and conclude with the promised characterization.

THEOREM 2. *If a Banach algebra A satisfies the conditions* (1) *and* (3), *then A is the completion of the direct join of its minimal closed two-sided ideals, each of which is a Banach algebra which satisfies* (1) *and* (3) *and is isomorphic and isometric to F(X) for some Banach space X. Conversely, F(X) is a topologically simple Banach algebra which satisfies* (1) *and* (3).

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