ON A CLASS OF PATHOLOGICAL FUNCTIONS

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We study the problem of approximating a function from a separable metric space D to a separable metric space Z by functions g from D to Z which have the property there is no set E of power $2^{\aleph_0}$ such that g is a homeomorphism of E onto $g(E)$. Theorem 1 asserts that any one to one function may be approximated by such a function. Numerous continuous functions may be approximated by such functions which, moreover, are periodic of preassigned period n (Theorem 4).

All functions mentioned are from a separable metric space into a separable metric space.

DEFINITION. A one to one function f of D into Z is a *dishomeomorphism* if D contains no subset E, of power $2^{\aleph_0}$, such that f is a homeomorphism of E onto $f(E)$.

The proof of the existence of a dishomeomorphism was given for the first time in [**2**].

By R is meant the set of real numbers, ordered in the natural manner. By a *linear* set is meant a subset of R.

DEFINITION. A one to one function f of a linear set E into R is a *dissimilarity transformation* (on E) if E contains no subset D, of power $2^{\aleph_0}$, such that f is a similarity transformation of D onto $f(D)$.

From the definitions there readily follows

LEMMA 1. *If E is of power $<2^{\aleph_0}$, then each one to one function f of E into Z is a dishomeomorphism. If E and $f(E)$ are linear sets, then f is also a dissimilarity transformation.*

It is easily seen that a necessary and sufficient condition that each one to one function f, of E into E, for which the power of the set $\{x \mid f(x)=x, x\in E\}$ is $<2^{\aleph_0}$, be a dissimilarity transformation, is that E have property A.[1]

The following known result will be used [**1**]:

LAVRENTIEFF'S THEOREM. *Let D be a subset of E. If f is a homeomorphism of D into a complete space C, then f can be extended to be a homeomorphism of a G_δ set B, containing D, into C.*

Let f be a function from E into D. We shall say that a function g

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[1] A set E has property A if E is a linear set of power $2^{\aleph_0}$ with the property that no two disjoint subsets of E, of power $2^{\aleph_0}$ each, are similar.

of E into D *ϵ-approximates* f if $d(f(x), g(x)) < \epsilon$ for each x in E, d being the metric on D.

LEMMA 2. *Let f be a mapping of D into a complete space Z such that for each element y in $E=f(D)$, $f^{-1}(y)$ is a denumerable set.*[2] *If each element of E is a c-condensation point of E,*[3] *then for each $\epsilon > 0$ there exists a dishomeomorphism g of D onto E which ϵ-approximates f.*

PROOF. For each element y in E let $S(y)$ denote the sphere in E, of radius $\epsilon/2$ and center y. Since each element of $S(y)$ is a c-condensation point of E, the power of $S(y)$ is $2^{\aleph_0}$. Furthermore, since $f^{-1}(y)$ is denumerable, $f^{-1}[S(y)]$ is of power $2^{\aleph_0}$.

Denote by F the set of all those couples (f, B), where B is a G_δ of D of power $2^{\aleph_0}$, and f is a homeomorphism of B into Z. Since D and Z are separable and of power $2^{\aleph_0}$ each, the power of the set F is $2^{\aleph_0}$. Well order the elements of D, E, and F into the sequences $\{u_\xi\}$, $\xi < \theta$, $\{v_\xi\}$, $\xi < \theta$, and $\{(f_\xi, B_\xi)\}$, $\xi < \theta$ respectively.[4] Denote by w_0 the element u_0. Let x_0 be the first element in the set $S(f(w_0)) - \{f_0(w_0)\}$. Define $g(u_0)$ to be x_0. Let z_0 be the first element in the set $E - \{x_0\}$ and y_0 the first element in $f^{-1}(S(z_0)) - \{w_0, f_0^{-1}(z_0)\}$. Define $g(y_0)$ to be z_0. Note that $d(f(w_0), g(w_0)) < \epsilon$ and $d(f(y_0), g(y_0)) < \epsilon$. Continuing by induction suppose that the elements w_ξ, $x_\xi = g(w_\xi)$, y_ξ, and $z_\xi = g(y_\xi)$ have been defined for each $\xi < \alpha$. Let w_α be the first element in the set

$$D - \{w_\xi, y_\xi \mid \xi < \alpha\} \tag{1}$$

and x_α the first element in the set

$$S(f(w_\alpha)) - [\{x_\xi, z_\xi \mid \xi < \alpha\} \cup \{f_\xi(w_\alpha) \mid \xi \leqq \alpha\}]. \tag{2}$$

Define $g(w_\alpha)$ to be x_α. Let z_α be the first element in the set

$$E - [\{x_\xi, z_\xi \mid \xi < \alpha\} \cup \{x_\alpha\}]. \tag{3}$$

Let y_α be the first element in

$$f^{-1}(S(z_\alpha)) - [\{w_\xi, y_\xi \mid \xi < \alpha\} \cup \{f_\xi^{-1}(z_\alpha) \mid \xi \leqq \alpha\} \cup \{w_\alpha\}]. \tag{4}$$

Since the set in the brackets is of power $< 2^{\aleph_0}$, y_α exists. Define $g(y_\alpha)$ to be z_α. Note that $d(f(w_\alpha), g(w_\alpha)) < \epsilon$ and $d(f(y_\alpha), g(y_\alpha)) < \epsilon$.

Clearly the function g is one to one. From (2) and (4) it follows

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[2] If f maps A into B, then by $f^{-1}(C)$, C being a subset of B, is meant $\{x \mid x \in A, f(x) \in C\}$.

[3] p is a c-condensation point of E if each open set containing p meets E in $2^{\aleph_0}$ elements.

[4] By θ is meant the smallest ordinal number whose power is $2^{\aleph_0}$.

that for $\alpha \geqq \xi$, $g(w_\alpha) \neq f_\xi(w_\alpha)$ and $g(y_\alpha) \neq f_\xi(y_\alpha)$. Hence

(5) the set $\{x \mid f_\xi(x) = g(x),\ x \in B_\xi\}$ is of power $< 2^{\aleph_0}$.

From (1) for each element u_ν of D, u_ν is in the set $\{w_\xi, y_\xi \mid \xi \leqq \nu\}$. From (3) for each element v_ν of E, v_ν is in the set $\{x_\xi, z_\xi \mid \xi \leqq \nu\}$. Consequently g is a one to one function of D onto E which ϵ-approximates f.

We now show that g is a dishomeomorphism. For suppose the contrary. Then there exists a subset H, of power $2^{\aleph_0}$, of D such that g is a dishomeomorphism of H onto $g(H)$. Let h be the function which is defined by $h(x) = g(x)$ for x in H. By Lavrentieff's Theorem, h may be extended to be a homeomorphism k of a G_δ set M, containing H, into E. This however contradicts (5) since k and g coincide on a set of power $2^{\aleph_0}$. Therefore g must be a dishomeomorphism. Q.E.D.

Suppose that f is a one to one mapping of Y into Z. Let E be the set of c-condensation points of $f(Y)$ which are in $f(Y)$ and let $G = f(Y) - E$. As is well known the power of G is $< 2^{\aleph_0}$, and each element of E is a c-condensation point of E. Let $D = f^{-1}(E)$. By Lemma 2, for $\epsilon > 0$ there exists a dishomeomorphism g of D onto E which ϵ-approximates f. For each element x in $f^{-1}(G) = Y - D$, let $g(x) = f(x)$. Now the following is easily seen.

LEMMA 3. *Let D be the union of two disjoint sets F and G. A necessary and sufficient condition that a one to one function f of D into E be a dishomeomorphism is that f be a dishomeomorphism of each of the sets F and G.*

In view of Lemma 3 and the preceding discussion we obtain

THEOREM 1. *For each one to one mapping f of D into E, and for each $\epsilon > 0$, there exists a dishomeomorphism g of D onto $f(D)$ which ϵ-approximates f.*

COROLLARY 1. *Each one to one function f of D into E is the limit of a uniformly convergent sequence of dishomeomorphisms of D onto $f(D)$.*

COROLLARY 2. *Let D be a subset of E and g a dishomeomorphism of D onto the subset F of G. Then g can be extended to be a dishomeomorphism of E onto G if and only if the two sets $E - D$ and $G - g(D)$ are of the same power.*

Suppose that f is a similarity transformation of a linear set E, of power $2^{\aleph_0}$, into R. The set D of points of discontinuity of f are enumerable. Thus f is a continuous similarity transformation of the set $E - D$, of power $2^{\aleph_0}$, onto $f(E - D) = C$. The function f^{-1} is a similarity transformation defined on C. Let B be the denumerable set of points of discontinuity of f^{-1} on C. Then f^{-1} is a continuous similarity transformation of the set $C-B$, of power $2^{\aleph_0}$, onto $f^{-1}(C-B)=A$. Consequently f is both a similarity transformation and a homeomorphism of A onto $f(A)$. From this it follows that if f is a dishomeomorphism of a linear set, then f is a dissimilarity transformation. Hence

COROLLARY 3. *For each one to one mapping of a linear set E into R, and for each $\epsilon>0$, there exists a function g, which is both a dishomeomorphism and a dissimilarity transformation, of E onto $f(E)$. Furthermore, g ϵ-approximates f.*

If, in Lemma 3 and Corollary 2, all sets are linear, then "dishomeomorphism" may be replaced by "dissimilarity transformation."

By a procedure quite analogous to Theorem 1 the following result may be proved.

THEOREM 2. *Let f be a mapping of D into itself such that for each element y in $E=f(D)$, $f^{-1}(y)$ is a denumerable set. If each element of E is a c-condensation point of E, then for each $\epsilon>0$ there exists a dishomeomorphism g of D onto E which ϵ-approximates f and also has the property that there is no element x in D and no positive integer n such that $f^n(x)=x$.*

COROLLARY. *Let f be a mapping of D into itself such that for each element y in $E=f(D)$, $f^{-1}(y)$ is a denumerable set. Suppose that there is no element x in D and positive integer n such that $f^n(x)=x$. Then there exists a dishomeomorphism g of D onto E which ϵ-approximates f and also has the property that there is no element x in D and no positive integer n such that $f^n(x)=x$.*

A function f of A into A which is not the identity is said to be of period two if $f^2(x)=x$ for every element x in A.[5] f is said to be strongly of period $n>0$ if $f^n(x)=x$ for each x in D, and for no $0<j<n$ does there exist an x so that $f^j(x)=x$.

THEOREM 3. *Let f be a one to one function of a nondenumerable space A into itself with the following property: For no odd integer >5 is there an element x in A such that $f^n(x)=x$ and $f^j(x)\neq x$ for each $j=1, 2, \cdots, n-1$. Let ϵ be any positive number which has the property that $d(x, f^2(x)) <\epsilon/2$ for each x in A. Then there exists a dishomeomorphism g of A onto A which is of period two and ϵ-approximates f.*

PROOF. Since A is a separable metric space there exists a denumer-

[5] For each function f, $f^0(x)=x$ and $f^{n+1}(x)=f[f^n(x)]$. Let $f^{-n}(x)=(f^n)^{-1}(x)$.

able set of spheres $\{S_n | n<\omega\}$, of diameter $<\epsilon/2$, which form a basis for the topology of A. Denote by K_n^m the set

$$K_n^m = \{x \mid x \in S_m, f(x) \in S_n\}.$$

Evidently $A = \bigcup_{m,n} K_n^m$. Relabel the K_n^m as B_i, $i<\omega$, i.e., $\{K_n^m | m, n<\omega\} = \{B_i | i<\omega\}$. Let $C_i = f(B_i)$. Let P_i and Q_i be the set of c-condensation points of B_i and C_i which are in B_i and C_i respectively. Let

$$U = \bigcup_i [(B_i - P_i) \cup (C_i - Q_i)]$$

and

$$V = \{f^r(x) \mid x \in U, r < \omega\} \cup \{f^{-r}(x) \mid x \in U, r < \omega\}.$$

As is easily seen, the power of V is $<2^{\aleph_0}$. Let $D_i = B_i - V$ and $E_i = C_i - V$. Thus $f(D_i) = E_i$. Notice that if x is in D_i, then each element of D_i is a c-condensation point of D_i (thus, if D_i is nonempty, then D_i is of power $2^{\aleph_0}$), and each element of E_i is a c-condensation point of E_i.

We now define g on V. Let y be any element of V and consider the sequence

$$(1) \qquad \cdots f^{-n}(y), \cdots, f^{-1}(y), y, f(y), \cdots, f^m(y), \cdots.$$

If for some n the element $f^{-1}[f^{-n}(y)]$ does not exist, then without loss of generality we may assume that $n=0$, i.e., $f^{-1}(y)$ does not exist. In this case (1) becomes

$$(2) \qquad y, f(y), \cdots, f^m(y), \cdots.$$

By assumption there is no odd integer $n>5$ and no element x in A such that $f^n(x) = x$ and $f^j(x) \neq x$ for each $j = 1, 2, \cdots, n-1$. If the elements in the sequence (1) or (2) are not all distinct, then there are 1, 3, 5, or an even number of distinct elements in (1) or (2). Let J_y be the set of elements in (1) or (2). For any two elements y and z in V either $J_y = J_z$ or else J_y and J_z are distinct. If the power of J_y is either infinite or even, let $g[f^{2n}(y)] = f^{2n+1}(y)$ and $g[f^{2n+1}(y)] = f^{2n}(y)$ for $n \geqq 0$, and let $g[f^{-2n}(y)] = f^{1-2n}(y)$ and $g[f^{1-2n}(y)] = f^{-2n}(y)$ for $n>0$. Suppose that J_y contains just three elements, y, $f(y)$, and $f^2(y)$, i.e., $f^3(y) = y$. Let $g(y) = y$, $g[f(y)] = f^2(y)$, and $g[f^2(y)] = f(y)$. On calculating $d(f(x), g(x))$ for x in J_y we have

$$d(f[f(y)], g[f(y)]) = 0 \text{ and } d(f[f^2(y)], g[f^2(y)]) = d(f^3(y), f(y)) < \epsilon/2.$$

Now $d(y, f^2(y)) < \epsilon/2$ and $d(f^2(y), f(y)) = d(f^2(y), f^4(y)) < \epsilon/2$. Thus

$$d[f(y), g(y)] = d(f(y), y) \leqq d(y, f^2(y)) + d(f^2(y), f(y)) < \epsilon/2 + \epsilon/2 = \epsilon.$$

Suppose that J_y contains just five elements, $y, f(y), f^2(y), f^3(y)$, and $f^4(y)$. Since $d(f(y), f^3(y))<\epsilon/2$ and $d(f^3(y), y)<\epsilon/2$, $d(y, f(y))<\epsilon$. Likewise $d(z, f(z))<\epsilon$ for each z in J_y. Define $g(y)=f(y)$, $g[f(y)]=f^3(y)$, $g[f^3(y)]=f(y)$, $g[f^2(y)]=f^4(y)$, and $g[f^4(y)]=f^2(y)$. For each of the five elements, $d(f(x), g(x))<\epsilon$. Suppose that J_y contains but one element. If $V\neq A$ let $g(y)=y$. Suppose that $V=A$, i.e., A is of power $<2^{\aleph_0}$. If, for some x, J_x contains more than one element, let $g(y)=y$. Suppose that $J_x=\{x\}$ for each x in V. Since V is nondenumerable, there exist two elements u and v in V such that $d(u, v)<\epsilon$. Let $g(u)=v$, $g(v)=u$, and $g(x)=x$ for each x in $V-\{u, v\}$. If $V=A$, then g is a well defined function of V onto V, of period two, which ϵ-approximates f. By Lemma 1, g is a dishomeomorphism. If $V\neq A$, then g is a function of period $\leqq 2$.

Suppose that A is of power $2^{\aleph_0}$. Let $W=A-V$. We modify the demonstration given in Theorem 1. Denote by F the set of those couples (f, B), where B is a G_δ of A of power $2^{\aleph_0}$, and f is a homeomorphism of B into A^*, A^* being the completion of A. Well order the elements of W and of F into the sequence $\{u_\xi\}$, $\xi<\theta$, and $\{(f_\xi, B_\xi)\}$, $\xi<\theta$, respectively. Suppose that the elements

$$w_\xi,\ x_\xi = g(w_\xi), \quad \text{and} \quad z_\xi = g(y_\xi)$$

have been defined for $\xi<\alpha$. Let w_α be the first element in

$$W - \{w_\xi, x_\xi, y_\xi, z_\xi \mid \xi < \alpha\}. \tag{3}$$

There exists a set D_i, say $D_{r(\alpha)}$, which contains w_α. Let x_α be the first element in the set

$$\begin{aligned} E_{r(\alpha)} - [\{w_\xi, x_\xi, y_\xi, z_\xi \mid \xi < \alpha\} \\ \cup \{w_\alpha\} \cup \{f_\xi(w_\alpha) \mid \xi \leqq \alpha\} \cup \{f_\xi^{-1}(w_\alpha) \mid \xi \leqq \alpha\}]. \end{aligned} \tag{4}$$

The element x_α certainly exists since $D_{r(\alpha)}$, thus $E_{r(\alpha)}$, is of power $2^{\aleph_0}$, whereas the set in the brackets is of power $<2^{\aleph_0}$. Define $g(w_\alpha)$ to be x_α and $g(x_\alpha)$ to be w_α. Since $f(w_\alpha)$ is in $E_{r(\alpha)}$ and the diameter of $E_{r(\alpha)}$ is $<\epsilon/2$, $d(f(w_\alpha), g(w_\alpha))<\epsilon$. Now there exists an element a_α in $D_{r(\alpha)}$ such that $f(a_\alpha)=x_\alpha$. Thus

$$\begin{aligned} d(f(x_\alpha), g(x_\alpha)) &= d(f^2(a_\alpha), w_\alpha) \leqq d(a_\alpha, w_\alpha) + d(a_\alpha, f^2(a_\alpha)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

Let z_α be the first element in the set

$$W - [\{w_\xi, x_\xi, y_\xi, z_\xi \mid \xi < \alpha\} \cup \{w_\alpha, x_\alpha\}]. \tag{5}$$

$f(z_\alpha)$ is in one of the sets D_i, say $D_{s(\alpha)}$. Let y_α be the first element in the set

$$D_{s(\alpha)} - [\{w_\xi, x_\xi, y_\xi, z_\xi \mid \xi < \alpha\} \cup \{w_\alpha, x_\alpha, z_\alpha\} \cup \{f_\xi(z_\alpha), f_\xi^{-1}(z_\alpha) \mid \xi \leqq \alpha\}]. \tag{6}$$

Define $g(y_\alpha)$ to be z_α and $g(z_\alpha)$ to be y_α. Evidently $d(f(z_\alpha), g(z_\alpha)) = d(f(z_\alpha), y_\alpha) < \epsilon$. Also

$$d(f(y_\alpha), g(y_\alpha)) = d(f(y_\alpha), z_\alpha) \leqq d(f(y_\alpha), f^2(z_\alpha)) + d(z_\alpha, f^2(z_\alpha)) < \epsilon.$$

The function g is well defined, of period two (since g is not the identity mapping), and ϵ-approximates f. From (3) and (5), g maps W onto W, thus A onto A. As in Theorem 1 we see that g is a dishomeomorphism. Q.E.D.

COROLLARY. *Each function, of period two, of a nondenumerable space A into itself is the limit of a uniformly convergent sequence of functions $\{f_n(x)\}$, each function being of period two, and each function being a dishomeomorphism of A onto A.*

If "nondenumerable" is removed from the hypothesis of Theorem 3, then the conclusion is no longer valid. For example, let A be the set of positive integers, f the identity, and $\epsilon = 1/3$. The only function which ϵ-approximates f is f itself. However f is not of period two.

The condition "$n > 5$" cannot be removed from Theorem 3. For example, consider a regular heptagon inscribed in the circle of radius one and center the origin, the vertices in clockwise order being x_1, x_6, x_4, x_2, x_7, x_5, and x_3. Let $A = \{x_i \mid i \leqq 7\} \cup M \cup N$, where $M = \{(4, y) \mid 0 \leqq y \leqq 1/2\}$ and $N = \{(41/10, y) \mid 0 \leqq y \leqq 1/2\}$. Let $\epsilon/2 = d(x_1, x_6) + 1/1000$. By trigonometry, $d(x_1, x_2) > \epsilon$. Let $f(x_7) = x_1$ and for $i < 7$ let $f(x_i) = x_{i+1}$. Let $f(4, y) = (41/10, y)$ and $f(41/10, y) = (4, y)$. As is easily seen, any function g, of period two, which ϵ-approximates f, maps a vertex onto a vertex. Hence for at least one k, $g(x_k) = x_k$. Then $d(f(x_k), g(x_k)) > \epsilon$, i.e., g goes not ϵ-approximate f.

In view of the previous example, an arbitrary function of a space D need not be ϵ-approximated by a function which is strongly of period n. However, we do have

THEOREM 4. *Suppose that each element of D is a c-condensation point of D. Let n be any integer > 1 and let f be a continuous function of D into itself such that for each x in D,* (i) *$d(x, f^n(x)) < \epsilon/2$, and* (ii) *$f^{-1}(x)$ is of power $< 2^{\aleph_0}$. Then there exists a function g, strongly of period n, which ϵ-approximates f.*

PROOF. Let $S(x)$ denote the sphere in D of radius ϵ and center x. Let F be the set of all pairs (h, B), where B is a G_δ of D and h is a homeomorphism of B into E, E being the completion of D. Well order the elements of D and F into the sequences $\{x_\xi\}$, $\xi<\theta$, and (f_ξ, B_ξ), $\xi<\theta$, respectively. For $\xi<\alpha$ suppose that u_i^ξ, $1\leqq i\leqq n$, have been defined so that $g(u_j^\xi)=u_{j+1}^\xi$ for $j<n$ and $g(u_n^\xi)=u_1^\xi$. Let u_1^α be the first element in $D-G_\alpha$, where $G_\alpha=\{u_j^\xi \mid \xi<\alpha, j\leqq n\}$. Let O_1^α be an open subset of $S(u_1^\alpha)$ containing u_1^α such that $f(O_1^\alpha)\subseteq S(f(u_1^\alpha))$. The continuity of f assures us that O_1^α exists. Since O_1^α is open, each element in O_1^α is a c-condensation point of O_1^α. The continuity of f and (ii) of the hypothesis imply that each point of $f(O_1^\alpha)$ is a c-condensation point of $f(O_1^\alpha)$. Suppose that O_i^α and u_i^α have been defined for $i<j<n$ so that O_i^α contains u_i^α, $f(O_i^\alpha)\subseteq S(f(u_i^\alpha))$, and each point of $f(O_i^\alpha)$ is a c-condensation point of $f(O_i^\alpha)$. Let u_j^α be the first element in

$$f(O_{j-1}^\alpha) - [G_\alpha \cup \{u_i^\alpha \mid i<j\} \cup \{f_\xi(u_{j-1}^\alpha) \mid \xi \leqq \alpha\}].$$

The power condition on the sets guarantee the existence of u_j^α. Let O_j^α, $O_j^\alpha\subseteq f(O_{j-1}^\alpha)$ be an open set, relative to $f(O_{j-1}^\alpha)$, containing u_j^α so that $f(O_j^\alpha)\subseteq S(f(u_{j-1}^\alpha))$. Since each element of $f(O_{j-1}^\alpha)$ is a c-condensation point of $f(O_{j-1}^\alpha)$ and O_j^α is open in $f(O_{j-1}^\alpha)$, O_j^α, thus $f(O_j^\alpha)$ have the same property. Let u_n^α be the first element in

$$f(O_{n-1}^\alpha) - [G_\alpha \cup \{u_i^\alpha \mid i \leqq n-1\} \cup \{f_\xi(u_{n-1}^\alpha), \overset{-1}{f_\xi}(u_1^\alpha) \mid \xi \leqq \alpha\}].$$

Let $g(u_i^\alpha)=u_{i+1}^\alpha$ for $i<n$ and $g(u_n^\alpha)=u_1^\alpha$.

By the induction, g is defined on all of D. Evidently g is strongly of period n. Being periodic, g maps D onto D. Repeating the argument used in Lemma 2 in conjunction with $f_\xi(u_j^\alpha)\neq g(u_j^\alpha)$ for $\alpha\geqq\xi$ and $j\leqq n$, it follows that g is a dishomeomorphism. For $i<n$, $g(u_i^\alpha)$ is in $S(f(u_i^\alpha))$, i.e., $d(f(u_i^\alpha), g(u_i^\alpha))<\epsilon$. Since $O_{i+1}^\alpha\subseteq f(O_i^\alpha)$ for $i<n-1$, there exists a z_α in O_1^α such that $f^{n-1}(z_\alpha)=u_n^\alpha$. Then

$$\begin{aligned} d(f(u_n^\alpha), g(u_n^\alpha)) &= d(f(u_n^\alpha), u_1^\alpha) \leqq d(u_1^\alpha, z_\alpha) \pm d(z_\alpha, f^n(z_\alpha)) \\ &< \epsilon/2 + \epsilon/2 = \epsilon, \end{aligned}$$

since $d(x, f^n(x))<\epsilon/2$ by hypothesis. Hence g ϵ-approximates f.

THEOREM 5. *For each infinite space D and each integer $n>1$ there exists a dishomeomorphism which is strongly of period n on D.*

PROOF. Let B be the set of c-condensation points of D which are in D. If D is of power $<2^{\aleph_0}$, let $A=D$. If D is of power $2^{\aleph_0}$, let $A=(D-B)\cup F$, where F is a denumerably infinite subset of D. In either case A is an infinite set of power $<2^{\aleph_0}$. Let A be the union of n disjoint sets A_i, each having the same power as A. This is possible since A is infinite. For $i<n$ let f_i be a one to one mapping of A_i onto

A_{i+1}, and let f_n be a one to one mapping of A_n onto A_1. For x in A_i let $f(x) = f_i(x)$. Since A is of power $< 2^{\aleph_0}$, f is a dishomeomorphism. Clearly f is strongly of period n. If $A = D$, then f is the desired function. Suppose that $A \neq D$. By Theorem 4, there exists a dishomeomorphism g, strongly of period n, on $D - A$, which ϵ-approximates the identity function. The function h, which coincides with f on A and coincides with g on $D - A$, is then the desired function.

We close with the following question: Given two separable metric spaces D and E of power $2^{\aleph_0}$ each, does there exist a one to one function f of D into E so that for each subset A of D, of power $2^{\aleph_0}$, A and $f(A)$ are not homeomorphic (not necessarily under f)?

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