ON SPACES FILLED UP BY CONTINUOUS COLLECTIONS
OF ATRIODIC CONTINUOUS CURVES

R. D. ANDERSON AND MARY-ELIZABETH HAMSTROM

Throughout this paper, $G$ will denote a nondegenerate continuous collection of atriodic continuous curves (i.e. arcs or simple closed curves) filling up a compact metric continuum $M$. As is well known, we may regard $G$ itself as a compact metric continuum, with the elements of the collection $G$ as the points of the space $G$ and with $G$, as a space, the image of $M$ under an open continuous mapping whose inverse sets are the elements of the collection $G$.

The principal results of this paper are the following theorems.

**Theorem I.** No closed, totally disconnected point set separates $M$.\(^1\)

**Theorem II.** If each element of $G$ is an arc, then no closed, totally disconnected point set separates any connected open subset of $M$.

**Theorem III.** If $G$ is a two-dimensional Cantor manifold,\(^2\) then $M$ is not separated by any rational curve.\(^3\)

As an immediate corollary of Theorem I, we have

**Theorem IV.** At no point of $M$ is the dimension of $M$ less than 2.

A well known result cited in [2, Theorem VI17, page 91] states, in effect, that if $\dim M - \dim G = k > 0$, then at least one element of $G$ has dimension not less than $k$. From Theorem IV, this result, and the fact that every atriodic continuous curve is one-dimensional, we obtain

**Theorem V.** If $G$ is a one-dimensional continuum, then $M$ is two-dimensional at each of its points.

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\(^1\) The theorem of [1] shows emphatically that Theorem I cannot be strengthened to the extent of deleting the condition that the continuous curves of $G$ be atriodic.

\(^2\) An $n$-dimensional Cantor manifold is a compact metric $n$-dimensional space which is not separated by any $(n-2)$-dimensional subspace.

\(^3\) A rational curve is a compact metric continuum $K$ such that each point of $K$ is contained in arbitrarily small neighborhoods relative to $K$ whose boundaries are countable. It is to be noted that a rational curve is not necessarily locally connected.

\(^4\) Eldon Dyer has recently obtained some general and interesting theorems about the dimension of $G$ if $M$ is $n$-dimensional and $G$ is a continuous collection of arcs (or dendrons).
We shall give a proof of Theorem I in some detail. The proofs of Theorems II and III are similar to that of Theorem I. We shall simply indicate the arguments for these.

A simple chain in $M$ is a finite collection $x_1, x_2, \ldots, x_n$ of open sets such that $x_i \cap x_j$ exists if and only if $|i - j| \leq 1$. The sets $x_1, x_2, \ldots, x_n$ are called the links of the chain. We note that if $s$ is an open interval of the arc $t$ which lies in $M$, there is an open subset $U$ of $M$ such that $U \cdot t = s$ and $\overline{U} \cdot t = s$. If $t$ is an arc and $\epsilon$ is any positive number, then there exists a simple chain $C$ covering $t$ each of whose links is of diameter less than $\epsilon$ and intersects $t$ in a connected set.

A special case of a result of J. H. Roberts [3, Theorem 2] states that if $E$ is a continuous collection of arcs filling up a compact metric space, there is a subcollection $E'$ of $E$ such that $E'$ is dense in $E$ and $E$ is equicontinuous\(^6\) at each element of $E'$. An argument suggested by that outlined in [3] yields a similar result in the case where $E$ is a continuous collection of atriadic continuous curves. We do not give this argument in detail but henceforth let $G'$ be a subcollection of $G$ at each element of which $G$ is equicontinuous.

Proof of Theorem I. Suppose, contrary to the statement of Theorem I that some closed, totally disconnected point set $T$ separates $M$ into sets $D_1$ and $D_2$. Then, since $M$ is connected and $G$ is continuous, some element $g_1$ of $G$ intersects each of $D_1$ and $D_2$ and, since $G'$ is dense in $G$, some element $g$ of $G'$ intersects each of $D_1$ and $D_2$. Let $h$, with endpoints $a_1$ and $a_2$ in $D_1$ and $D_2$ respectively, be an arc in $g$. For each $i, i = 1, 2$, let $A_i$ be an open set containing $a_i$ with $\overline{A_i}$ contained in $D_i$. Let $C: c_1, \ldots, c_6$ be a simple chain covering $h$ such that (1) $c_1 - c_1 \cdot c_2$ contains $A_1$, (2) $c_6 - c_6 \cdot c_6$ contains $A_2$, (3) $c_1 + c_2$ is a subset of $D_1$, (4) $c_4 + c_5$ is a subset of $D_2$, (5) for $i = 1, \ldots, 5, c_i \cdot h$ is connected, and (6) for $i = 2, 3, 4, c_i$ and $g - h$ are mutually exclusive. Clearly such a chain exists. Let $U$ containing $g$ be an open subcollection of $G$ such that each element $u$ of $U$ contains an arc $u_x$ with endpoints in $A_1$ and $A_2$ respectively such that (1) $u_x$ is covered by $C$, (2) $u - u_x$ does not intersect $\overline{c_5}$, and (3) $u_x$ does not contain two disjoint arcs each intersecting $A_1$ and $c_3, A_2$ and $c_3$, or $c_2$ and $c_4$. From the equicontinuity of $G$ at $g$ it follows that such a set $U$ exists.

Let $G$ containing $g$ be a nondegenerate closed connected subcollection of $U$ and let $\overline{H}$ containing $h$ be a collection of arcs in one-to-one correspondence with $G$ such that each element of $\overline{H}$ is contained in $\overline{H}$.

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\(^6\) The collection $G$ as above is said to be equicontinuous at the element $g$ of $G$ provided that for any $\epsilon > 0$ and any point $p$ of $g$, there exists a $\delta > 0$ such that if $x$ and $y$ are points of the same element $g'$ of $G$ and are each within $\delta$ of $p$, then there is an arc in $g'$ containing $x + y$ and of diameter less than $\epsilon$. 

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the corresponding element of $\tilde{G}$, is covered by $C$, and has its endpoints in $A_1$ and $A_2$ respectively. Let $\tilde{H}$ be topologized so as to be homeomorphic with $\tilde{G}$ under the correspondence above.

Let $g'$ be an element of $\tilde{G}$ distinct from $g$ and let $h'$ be the corresponding element of $\tilde{H}$. Let $W$ be an open set in $c_3-(c_2+c_4)\cdot c_3$ such that $\tilde{W}-W$ does not intersect $T$, $W$ contains $T\cdot h$, and $W$ does not intersect $h'$. Let $Z_1$ and $Z_2$ be the subsets of $\tilde{W}-W$ in $D_1$ and $D_2$ respectively. Each of $Z_1$ and $Z_2$ is closed.

For each element $k$ of $\tilde{H}$, let $N(k)$ be the collection of those components of $k\cdot W$ having limit points in each of $Z_1$ and $Z_2$. Let $n(k)$ be the number (mod 2) of elements in $N(k)$. Let $\tilde{H}_0$ be the collection of all elements $k$ of $\tilde{H}$ for which $n(k) = 0$. The collection $\tilde{H}_0$ contains $h'$ and $\tilde{H}-\tilde{H}_0$ contains $h$. We wish to show that each of $\tilde{H}_0$ and $\tilde{H}-\tilde{H}_0$ is open and hence that $\tilde{G}$ is not connected—a contradiction.

Let $\tilde{h}$ be any element of $\tilde{H}$. There exists a simple chain $C(\tilde{h})$ covering $\tilde{h}$ such that (1) each link of $C(\tilde{h})$ is a subset of a link of $C$ and intersects $\tilde{h}$ and (2) no link of $C(\tilde{h})$ intersects (a) each of $D_1$ and $D_2$ but not $T$, (b) each of $Z_1$ and $Z_2$, or (c) each of $\tilde{W}$ and $T-T\cdot W$.

Let $X(\tilde{h})$ be the collection of links of $C(\tilde{h})$ intersecting $\tilde{W}$. Let $Y(\tilde{h})$ be the collection of all maximal simple chains whose links are elements of $X(\tilde{h})$. If $k$ is any element of $\tilde{H}$ containing a subarc $k'$ with endpoints in $A_1$ and $A_2$ and with $k'$ covered by $C(\tilde{h})$, then $n(k) = n(\tilde{h})$. This follows from the fact that each element of $N(k)$ is covered by exactly one element of $Y(\tilde{h})$ and the number of elements of $N(k)$ in an element $y$ of $Y(\tilde{h})$ is 1 or 0 (mod 2) according as the end links of $y$ do or do not intersect different sets of the two sets $Z_1$ and $Z_2$. Hence $\tilde{H}_0$ and $\tilde{H}-\tilde{H}_0$ are each open and Theorem I is proved.

**Indication of Proof of Theorem II.** Suppose, contrary to Theorem II that there exists a connected open set $D$ in $M$ separated by a closed and totally disconnected set $T$. Let $(D_1, D_2)$ be a separation of $D$ by $T$. Then some element of $G$ must contain an arc in $D$ with endpoints in $D_1$ and $D_2$ respectively. Since $G$ is a collection of arcs, some element of $G'$ must also contain an arc in $D$ with endpoints in $D_1$ and $D_2$ respectively and the argument is essentially reduced to that for Theorem I. We note that if the elements of $G$ are not restricted to being arcs, then there exist simple examples without such an element of $G'$ existing and with an open set separated by a point.

**Indication of Proof of Theorem III.** Suppose $G$ is a two-dimensional Cantor manifold and $M$ is separated by a rational curve $J$ into the two mutually separated sets $D_1$ and $D_2$. Clearly some element of $G$ must intersect each of $D_1$ and $D_2$ for otherwise the set of those elements of $G$ lying completely in $J$ must exist and be closed.
and in $G$ must separate $G$. But by Theorem I this set is 0-dimen-
sional, a contradiction. Hence some element of $G$ intersects each of
$D_1$ and $D_2$, and thus some element of $G'$ contains an arc $h$ with end-
points in $D_1$ and $D_2$ respectively. But by an argument similar to that
used in the proof of Theorem I we can exhibit an open subcollection
$U$ of $G$ with $U$ containing $g$ and with $\overline{U} - U$ countable, a contradic-
tion.

REFERENCES

2. Witold Hurewicz and Henry Wallman, Dimension theory, Princeton University
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THE UNIVERSITY OF PENNSYLVANIA AND
GOUCHER COLLEGE

A NOTE ON BASIC SETS OF HOMOGENEOUS
HARMONIC POLYNOMIALS

E. P. MILES, JR. AND E. WILLIAMS

For any set of non-negative integers $(b_j)$ such that $b_1 \leq 1$ and
$\sum_{j=1}^{k} b_j = n$, let

$$H_{b_1 \ldots b_k}^n = \sum (-1)^{[a_{1/2}]} \frac{n!}{\prod_{j=1}^{k} a_j!} \cdot \frac{\left[ \frac{a_1}{2} \right]!}{\prod_{j=2}^{k} \left( \frac{b_j - a_j}{2} \right)!} \prod_{j=1}^{k} x_j^{a_j}$$

where the summation is extended over all $(a_j)$ such that,

(a) $a_j \equiv b_j \mod 2$, $j = 1, 2, \ldots, k$,
(b) $\sum_{j=1}^{k} a_j = n$,
(c) $a_j \leq b_j$, $j = 2, 3, \ldots, k$.

The polynomials (1) were shown by the authors to form a basic set
of homogeneous harmonic polynomials in $k$ variables [1].

It is easily seen that the following differential recursion formulas
hold for these polynomials:

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genous polynomials in $k$ variables. II; received by the editors November 29, 1954.

1 Numbers in brackets refer to bibliography at the end of the paper.