ON SPACES FILLED UP BY CONTINUOUS COLLECTIONS OF ATRIODIC CONTINUOUS CURVES

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Throughout this paper, G will denote a nondegenerate continuous collection of atriodic continuous curves (i.e. arcs or simple closed curves) filling up a compact metric continuum M. As is well known, we may regard G itself as a compact metric continuum, with the elements of the collection G as the points of the space G and with G, as a space, the image of M under an open continuous mapping whose inverse sets are the elements of the collection G.

The principal results of this paper are the following theorems.

THEOREM I. *No closed, totally disconnected point set separates M.*[1]

THEOREM II. *If each element of G is an arc, then no closed, totally disconnected point set separates any connected open subset of M.*

THEOREM III. *If G is a two-dimensional Cantor manifold,*[2] *then M is not separated by any rational curve.*[3]

As an immediate corollary of Theorem I, we have

THEOREM IV.[4] *At no point of M is the dimension of M less than* 2.

A well known result cited in [2, Theorem VI7, page 91] states, in effect, that if $\dim M - \dim G = k > 0$, then at least one element of G has dimension not less than k. From Theorem IV, this result, and the fact that every atriodic continuous curve is one-dimensional, we obtain

THEOREM V. *If G is a one-dimensional continuum, then M is two-dimensional at each of its points.*

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[1] The theorem of [1] shows emphatically that Theorem I cannot be strengthened to the extent of deleting the condition that the continuous curves of G be atriodic.

[2] An n-dimensional Cantor manifold is a compact metric n-dimensional space which is not separated by any $(n-2)$-dimensional subspace.

[3] A rational curve is a compact metric continuum K such that each point of K is contained in arbitrarily small neighborhoods relative to K whose boundaries are countable. It is to be noted that a rational curve is not necessarily locally connected.

[4] Eldon Dyer has recently obtained some general and interesting theorems about the dimension of G if M is n-dimensional and G is a continuous collection of arcs (or dendrons).

We shall give a proof of Theorem I in some detail. The proofs of Theorems II and III are similar to that of Theorem I. We shall simply indicate the arguments for these.

A simple chain in M is a finite collection $x_1, x_2, \cdots, x_n$ of open sets such that $x_i \cdot x_j$ exists if and only if $|i-j| \leqq 1$. The sets $x_1, x_2, \cdots, x_n$ are called the links of the chain. We note that if s is an open interval of the arc t which lies in M, there is an open subset U of M such that $U \cdot t = s$ and $\overline{U} \cdot \bar{t} = \bar{s}$. If t is an arc and ϵ is any positive number, then there exists a simple chain C covering t each of whose links is of diameter less than ϵ and intersects t in a connected set.

A special case of a result of J. H. Roberts [**3**, Theorem 2] states that if E is a continuous collection of arcs filling up a compact metric space, there is a subcollection E' of E such that E' is dense in E and E is equicontinuous[5] at each element of E'. An argument suggested by that outlined in [**3**] yields a similar result in the case where E is a continuous collection of atriodic continuous curves. We do not give this argument in detail but henceforth let G' be a subcollection of G at each element of which G is equicontinuous.

Proof of Theorem I. Suppose, contrary to the statement of Theorem I that some closed, totally disconnected point set T separates M into sets D_1 and D_2. Then, since M is connected and G is continuous, some element g_1 of G intersects each of D_1 and D_2 and, since G' is dense in G, some element g of G' intersects each of D_1 and D_2. Let h, with endpoints a_1 and a_2 in D_1 and D_2 respectively, be an arc in g. For each i, $i=1, 2$, let A_i be an open set containing a_i with $\overline{A}_i$ contained in D_i. Let C: $c_1, \cdots, c_5$ be a simple chain covering h such that (1) $c_1 - c_1 \cdot c_2$ contains A_1, (2) $c_5 - c_4 \cdot c_5$ contains A_2, (3) $\bar{c}_1 + \bar{c}_2$ is a subset of D_1, (4) $\bar{c}_4 + \bar{c}_5$ is a subset of D_2, (5) for $i=1, \cdots, 5$, $\bar{c}_i \cdot h$ is connected, and (6) for $i=2, 3, 4$, c_i and $g-h$ are mutually exclusive. Clearly such a chain exists. Let U containing g be an open subcollection of G such that each element u of U contains an arc u_x with endpoints in A_1 and A_2 respectively such that (1) u_x is covered by C, (2) $u - u_x$ does not intersect $\bar{c}_3$, and (3) u_x does not contain two disjoint arcs each intersecting A_1 and c_3, A_2 and c_3, or c_2 and c_4. From the equicontinuity of G at g it follows that such a set U exists.

Let $\tilde{G}$ containing g be a nondegenerate closed connected subcollection of U and let $\tilde{H}$ containing h be a collection of arcs in one-to-one correspondence with $\tilde{G}$ such that each element of $\tilde{H}$ is contained in

[5] The collection G as above is said to be equicontinuous at the element g of G provided that for any $\epsilon > 0$ and any point p of g, there exists a $\delta > 0$ such that if x and y are points of the same element g' of G and are each within δ of p, then there is an arc in g' containing $x+y$ and of diameter less than ϵ.

the corresponding element of $\tilde{G}$, is covered by C, and has its endpoints in A_1 and A_2 respectively. Let $\tilde{H}$ be topologized so as to be homeomorphic with $\tilde{G}$ under the correspondence above.

Let g' be an element of $\tilde{G}$ distinct from g and let h' be the corresponding element of $\tilde{H}$. Let W be an open set in $c_3-(c_2+c_4)\cdot c_3$ such that $\overline{W}-W$ does not intersect T, W contains $T\cdot h$, and W does not intersect h'. Let Z_1 and Z_2 be the subsets of $\overline{W}-W$ in D_1 and D_2 respectively. Each of Z_1 and Z_2 is closed.

For each element k of $\tilde{H}$, let $N(k)$ be the collection of those components of $k\cdot W$ having limit points in each of Z_1 and Z_2. Let $n(k)$ be the number (mod 2) of elements in $N(k)$. Let $\tilde{H}_0$ be the collection of all elements k of $\tilde{H}$ for which $n(k)=0$. The collection $\tilde{H}_0$ contains h' and $\tilde{H}-\tilde{H}_0$ contains h. We wish to show that each of $\tilde{H}_0$ and $\tilde{H}-\tilde{H}_0$ is open and hence that $\tilde{G}$ is not connected—a contradiction.

Let $\tilde{h}$ be any element of $\tilde{H}$. There exists a simple chain $C(\tilde{h})$ covering $\tilde{h}$ such that (1) each link of $C(\tilde{h})$ is a subset of a link of C and intersects $\tilde{h}$ and (2) no link of $C(\tilde{h})$ intersects (a) each of D_1 and D_2 but not T, (b) each of Z_1 and Z_2, or (c) each of $\overline{W}$ and $T-T\cdot W$.

Let $X(\tilde{h})$ be the collection of links of $C(\tilde{h})$ intersecting $\overline{W}$. Let $Y(\tilde{h})$ be the collection of all maximal simple chains whose links are elements of $X(\tilde{h})$. If k is any element of $\tilde{H}$ containing a subarc k' with endpoints in A_1 and A_2 and with k' covered by $C(\tilde{h})$, then $n(k)=n(\tilde{h})$. This follows from the fact that each element of $N(k)$ is covered by exactly one element of $Y(\tilde{h})$ and the number of elements of $N(k)$ in an element y of $Y(\tilde{h})$ is 1 or 0 (mod 2) according as the end links of y do or do not intersect different sets of the two sets Z_1 and Z_2. Hence $\tilde{H}_0$ and $\tilde{H}-\tilde{H}_0$ are each open and Theorem I is proved.

Indication of proof of Theorem II. Suppose, contrary to Theorem II that there exists a connected open set D in M separated by a closed and totally disconnected set T. Let (D_1, D_2) be a separation of D by T. Then some element of G must contain an arc in D with endpoints in D_1 and D_2 respectively. Since G is a collection of arcs, some element of G' must also contain an arc in D with endpoints in D_1 and D_2 respectively and the argument is essentially reduced to that for Theorem I. We note that if the elements of G are not restricted to being arcs, then there exist simple examples without such an element of G' existing and with an open set separated by a point.

Indication of proof of Theorem III. Suppose G is a two-dimensional Cantor manifold and M is separated by a rational curve J into the two mutually separated sets D_1 and D_2. Clearly some element of G must intersect each of D_1 and D_2 for otherwise the set of those elements of G lying completely in J must exist and be closed and in G must separate G. But by Theorem I this set is 0-dimensional, a contradiction. Hence some element of G intersects each of D_1 and D_2, and thus some element of G' contains an arc h with end-points in D_1 and D_2 respectively. But by an argument similar to that used in the proof of Theorem I we can exhibit an open subcollection U of G with U containing g and with $\overline{U}-U$ countable, a contradiction.

References

1. R. D. Anderson, *Continuous collections of continuous curves*, Duke Math. J. vol. 21 (1954) pp. 363–368.

2. Witold Hurewicz and Henry Wallman, *Dimension theory*, Princeton University Press, 1948.

3. J. H. Roberts, *Collections filling a plane*, Duke Math. J. vol. 2 (1936) pp. 10–19.

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A NOTE ON BASIC SETS OF HOMOGENEOUS HARMONIC POLYNOMIALS

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For any set of non-negative integers (b_j) such that $b_1 \leqq 1$ and $\sum_{j=1}^{k} b_j = n$, let

$$(1) \qquad H^{n}_{b_1 \cdots b_k} = \sum (-1)^{[a_1/2]} \frac{n!}{\prod_{j=1}^{k} a_j!} \cdot \frac{\left[\dfrac{a_1}{2}\right]!}{\prod_{j=2}^{k} \left(\dfrac{b_j - a_j}{2}\right)!} \prod_{j=1}^{k} x_j^{a_j}$$

where the summation is extended over all (a_j) such that,

(a) $a_j \equiv b_j \bmod 2, j = 1, 2, \cdots, k,$

(b) $\sum_{j=1}^{k} a_j = n,$

(c) $a_j \leqq b_j, j = 2, 3, \cdots, k.$

The polynomials (1) were shown by the authors to form a basic set of homogeneous harmonic polynomials in k variables [1].[1]

It is easily seen that the following differential recursion formulas hold for these polynomials:

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[1] Numbers in brackets refer to bibliography at the end of the paper.