RIGHT ALTERNATIVE RINGS OF CHARACTERISTIC TWO

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1. **Introduction.** Right alternative rings have recently been investigated by Skornyakov, Kleinfeld, and the author. Skornyakov [3][1] showed that a right alternative division ring of characteristic not two is alternative. The author, in [2], extended this result by proving that a right alternative division ring of characteristic two is alternative if (and only if) it satisfies

$$w(xy\cdot x) = (wx\cdot y)x \tag{1.1}$$

for all w, x, y and showed by example that (1.1) can fail to hold. Prior to this, Kleinfeld [1] generalized the Skornyakov theorem in another direction by assuming only the absence of one sort of nilpotent element. We now specify Kleinfeld's result in detail.

Let F be the free nonassociative ring generated by x_1 and x_2 and suppose that R is any right alternative ring. Kleinfeld calls t, u, v in R an *alternative triple* if (i) there exist elements $\alpha[x_1, x_2]$, $\beta[x_1, x_2]$, $\gamma[x_1, x_2]$ in F and elements r_1, r_2 in R such that $t=\alpha[r_1, r_2]$, $u=\beta[r_1, r_2]$, $v=\gamma[r_1, r_2]$ and (ii) if s_1 and s_2 are elements from an arbitrary alternative ring, and if $t'=\alpha[s_1, s_2]$, $u'=\beta[s_1, s_2]$, $v'=\gamma[s_1, s_2]$, then $(t', u', v')=0$. The ring R is said to have *property* (P) if t, u, v an alternative triple in R and $(t, u, v)^2=0$ imply $(t, u, v)=0$. By the definition of an alternative triple, an alternative ring has property (P). Kleinfeld's result is the converse, assuming characteristic not two; that is, a right alternative ring of characteristic not two is alternative if (and only if) it has property (P).

We herein extend this line of investigation by proving that a right alternative ring of characteristic two, satisfying (1.1), is alternative if (and only if) it has property (P). The methods are mainly those used in [2], coupled with two essential lemmas (numbered 4 and 5 in our paper) due to Kleinfeld. Following [2], we say that R is *strongly right alternative* if R is a right alternative ring satisfying (1.1). Throughout the paper, R will always denote such a ring, with the additional hypothesis that R have characteristic two.

2. **Previous results.** We begin with the following definition due to Skornyakov [3]. Let a and b be fixed elements of R. Then we shall

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[1] Numbers in brackets refer to the bibliography at the end of the paper.

denote by $u(a, b)$ the set of all elements x in R such that $xa \cdot b = x \cdot ba$. We observe that $u(a, b)$ is closed under addition.

LEMMA 1. *The following identities hold in* R:

$$(2.1) \qquad (w, x, xy) = (w, x, y)x,$$

$$(2.2) \qquad (w, x, yz) + (w, y, xz) = (w, x, z)y + (w, y, z)x,$$

$$(2.3) \qquad (w, x^2, y) = (w, x, (x, y)),$$

$$(2.4) \qquad ((w, x, y), x, y) = (w, x, y)(x, y),$$

$$(2.5) \qquad (wx, y, z) = w(x, y, z) + (w, y, z)x + (w, x, (y, z)).$$

LEMMA 2. *x is in $u(a, b)$ if and only if* $(x, a, b) = x(a, b)$.

LEMMA 3. *If x is in $u(a, b)$ and x is in $u(a, ba)$, then* $x(a, a, b) = 0$.

LEMMA 4. (x, a, b) *and* $(x, a, b)a$ *are in* $u(a, b)$.

LEMMA 5. *If both y and xy are in $u(a, b)$, then* $(x, a, b)y = 0$.

Proofs of Lemmas 1–3 may be found in [**3**], and proofs of Lemmas 4 and 5 in [**1**].

As in [**2**], we define the mapping π (of R into R) by $x\pi = (x, a, b)$, for fixed a, b in R. Then (2.5) may be written

$$(2.6) \qquad (wx)\pi = w \cdot x\pi + w\pi \cdot x + (w, x, (a, b)).$$

3. The main theorem.

We henceforth assume that R, a strongly right alternative ring of characteristic two, has property (P).

LEMMA 6. $(a, b) = 0$ *implies* $(a, a, b) = 0$.

PROOF. Assuming $(a, b) = 0$, we have that $(xa, a, b) = x(a, a, b) + (x, a, b)a$, using (2.5). Lemma 4 can be invoked to show that $x(a, a, b)$ is in $u(a, b)$. But (a, a, b) is also, so that, by Lemma 5, $(x, a, b)(a, a, b) = 0$. Setting $x = a$ and using property (P) proves the lemma.

For convenience, we set $c = (a, a, b)$, $d = (a, b)$, and $e = (d, a, d)$. This enables us to state

LEMMA 7. (i) $d\pi = (d, a, b) = 0$, (ii) $(d, a, x) = ((a, x), a, b)$, (iii) $cd = dc$, (iv) $(c, c, d) = (d, d, c) = 0$.

PROOF. (i), (ii), and (iii) are proved in [**1**, Lemmas 5 and 7] since Kleinfeld does not use the assumption on the characteristic until he reaches his equation (4). Then (iv) follows from Lemma 6.

LEMMA 8. *For arbitrary u in $u(a, b)$,* $(d, c, u) = eu$.

PROOF. Observing that $u\pi = ud$, and $a\pi = c$, we compute $(d, a, u)\pi$ and obtain

$$(3.1) \qquad (d, a, u)\pi = (d, a, u)d + (d, c, u) + eu.$$

However, Lemma 7(ii) shows that (d, a, u) is in $u(a, b)$ and thus the lemma follows from Lemma 2 and (3.1).

LEMMA 9. $e = 0$.

PROOF. The first three sentences in the proof of Lemma 7 in [1] show that $ec = 0$. Hence $(de \cdot c)e = 0$, using (1.1). We now apply Lemma 8 with $u = e$ and get $e^2 = (d, e, c)$ so that $e^2 = de \cdot c$. Hence[2] $e^3 = 0$, so $e^4 = 0$. But property (P) implies $e^2 = 0$, and, again, $e = 0$.

LEMMA 10. *(c, a, d) is in $u(a, b)$.*

PROOF. As in the proof of Lemma 8, we compute $(x, a, d)\pi$ and obtain

$$(3.2) \quad (x, a, d)\pi = (x, a, d)d + ((x, a, b), a, d) + (x, c, d) + (x, ad, d).$$

Substituting $x = c$ in (3.2) gives $(c, a, d)\pi = (c, a, d)d + \theta$, where $\theta = (cd, a, d) + (c, d, ad) = ce + (c, a, d)d + (c, d, (a, d)) + (c, d, ad)$. However, $(c, d, ad) = (c, a, d^2) + (c, a, d)d = (c, d, (a, d)) + (c, a, d)d$, by (2.2) and (2.3). Hence $\theta = 0$ and the proof is complete.

We can now prove our main result.

THEOREM. *Let R be a strongly right alternative ring of characteristic two. Then R is alternative if and only if it has property* (P).

PROOF. The necessity is obvious. For the sufficiency, we begin by showing that c^2 is in $u(a, b)$. Indeed, $(ca, a, b) = c^2 + cd \cdot a + (c, a, d)$. However, Lemma 7(ii) shows that (d, a, c) is in $u(a, b)$. But $d \cdot ca$ is in so that $dc \cdot a = cd \cdot a$ is in, and (c, a, d) is in by Lemma 10. Hence c^2 is in $u(a, b)$. However c is also in $u(a, b)$, and thus, using Lemma 5, $cd \cdot c = c^2 d = 0$. But cd is in $u(a, b)$ and another application of Lemma 5 gives $(cd)^2 = 0$, from which $cd = 0$.

Now[3] $((a, a, b), a, b) = cd = 0$ and linearization yields $((a, a, x), a, b) = (c, a, x)$. Put $x = ab$ and obtain that $(ca, a, b) = 0$. This implies that $c^2 = (c, a, ba)$ and therefore c^2 is in $u(a, ba)$. Lemma 3 yields $c^3 = 0$, from which $c^4 = 0$, and, using property (P), $c^2 = 0$, $c = 0$.

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[2] It is an easy matter to verify that a strongly right alternative ring of arbitrary characteristic is power-associative. Therefore, powers of a single element are well-defined and we may write e^3, e^4, etc. without ambiguity.

[3] The last paragraph of our proof is the same as that in [1], but we repeat the few lines here for completeness.

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ON AN ITERATIVE PROCEDURE FOR OBTAINING THE PERRON ROOT OF A POSITIVE MATRIX

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1. Introduction. The purpose of this paper is to present a new iterative procedure for obtaining the characteristic root of largest absolute value of a positive matrix.

The origin of the method is as follows. There is a result of von Neumann [**7**], a generalization of his fundamental min-max theorem in the theory of games [**8**], to the effect that

$$\text{(1)} \qquad \operatorname*{Min}_{y} \operatorname*{Max}_{x} \frac{(x, Ay)}{(x, By)} = \operatorname*{Max}_{x} \operatorname*{Min}_{y} \frac{(x, Ay)}{(x, By)}$$

where the variation is over the region defined by

$$\text{(2)} \qquad R: \begin{array}{ll} \text{(a)} \ x_i \geqq 0, & \sum_{i=1}^{n} x_i = 1, \\ \text{(b)} \ y_i \geqq 0, & \sum_{i=1}^{n} y_i = 1, \end{array}$$

and it is assumed that B has the property that

$$(x, By) \geqq b > 0$$

for all $(x, y) \in R$.

It was observed by Shapley [**6**] that this result can be obtained as a by-product of the theory of "games of survival," cf. [**1**; **5**; **6**], which requires only the fundamental min-max theorem, by considering the equation for λ,

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