k ranges over the positive integers, are distinct, so b has infinitely many distinct conjugates.

Thus (c) holds in G; it is well known that G satisfies (b).

Bibliography

1. R. H. Bruck, *Difference sets in a finite group*, Trans. Amer. Math. Soc. vol. 78 (1955) pp. 464–481.
2. M. Hall, *Cyclic projective planes*, Duke Math. J. vol. 14 (1947) pp. 1079–1090.

The University of Wisconsin

MAXIMAL SUBALGEBRAS OF GROUP-ALGEBRAS

JOHN WERMER

A closed subalgebra of a Banach algebra is called *maximal* if it is not contained in any larger proper closed subalgebra. Let G be a discrete abelian topological group and L its group-algebra, i.e. L is the Banach algebra of functions f on G with $\sum_{\lambda\in G} |f(\lambda)| < \infty$ and multiplication defined as convolution. What are the maximal subalgebras of L? The complete answer is not known even when G is the group of integers.

Here we assume that G is ordered. Let G^+ be the semi-group of non-negative elements of G and L^+ the subset of L consisting of functions which vanish outside of G^+. Then L^+ is a proper closed subalgebra of L.

Theorem 1.[1] *L^+ is a maximal subalgebra of L if and only if the ordering of G is archimedean.*

Proof. Suppose the ordering is non-archimedean. Then we can find a, b in G^+ with $na<b$ for $n=1, 2, \cdots$. Consider the set G_1 of all elements of G of the form $g^+ + n(-a)$, where $n=0, 1, 2, \cdots$ and g^+ is in G^+. Clearly G_1 is a semi-group containing G^+ and also $-a$ is in G_1 and $-b$ is not in G_1. Let L_1 be the closed subalgebra of L consisting of all functions vanishing outside G_1. Then L_1 lies properly between L^+ and L, whence L^+ is not maximal.

Suppose now that the ordering of G is archimedean. Let $\mathfrak{A}'$ be a proper closed subalgebra of L with L^+ included in $\mathfrak{A}'$. We shall show $\mathfrak{A}'=L^+$.

Let E_λ be the function in L with $E_\lambda(g)=0$, $g\neq\lambda$, $E_\lambda(\lambda)=1$. Then

---

Received by the editors December 27, 1954.

[1] A proof of this theorem has also been found by I. M. Singer. See the note below.

for some $\lambda_0>0$, $E_{-\lambda_0}$ is not in $\mathfrak{A}'$. For $\mathfrak{A}'$ contains L^+ and so E_λ is in $\mathfrak{A}'$, for $\lambda>0$. If also all E_λ with $\lambda<0$ were to belong to $\mathfrak{A}'$, then $\mathfrak{A}'$ should equal L. Thus for some $\lambda_0>0$, E_{λ_0} is in $\mathfrak{A}'$ and its inverse $E_{-\lambda_0}$ is not. By a basic result of Gelfand there hence exists a (linear) multiplicative functional χ on $\mathfrak{A}'$ with $\chi(E_{\lambda_0})=0$.

Consider any $\lambda_1>0$. Then since the ordering is archimedean, there is a positive integer k with $-\lambda_0+k\lambda_1>0$. Hence $E_{-\lambda_0+k\lambda_1}$ is in $\mathfrak{A}'$. We therefore get

$$(\chi(E_{\lambda_1}))^k = \chi(E_{k\lambda_1}) = \chi(E_{\lambda_0})\chi(E_{-\lambda_0+k\lambda_1}) = 0$$

and so $\chi(E_{\lambda_1})=0$ for all $\lambda_1>0$.

L admits the involution

$$f \to f^* \quad \text{where} \quad f^*(\lambda) = \overline{f(-\lambda)}.$$

Let $\mathfrak{A}''$ be the closed subalgebra of $\mathfrak{A}'$ generated by the self-adjoint elements in $\mathfrak{A}'$. Then $\mathfrak{A}''$ is a self-adjoint subalgebra of $\mathfrak{A}'$ and contains the unit element. The functional χ restricted to $\mathfrak{A}''$ defines a multiplicative functional on $\mathfrak{A}''$. Now a theorem due to Silov [1] asserts that a multiplicative functional defined on a closed self-adjoint subalgebra of a Banach algebra may be extended to a multiplicative functional defined on the whole Banach algebra. We apply this theorem to extend χ from $\mathfrak{A}''$ to a multiplicative functional χ_0 defined on all of L.

We claim that if an element f of $\mathfrak{A}'$ has the form $f=\sum_{\lambda<0} a_\lambda E_\lambda$, then $\chi_0(f)=0$. For $f^*=\sum_{\lambda<0}\bar{a}_\lambda E_{-\lambda}$ is in L^+ and so is in $\mathfrak{A}'$. Hence f and f^* are in $\mathfrak{A}'$, whence f and f^* are in $\mathfrak{A}''$. Then

$$\chi_0(f^*) = \chi(f^*) = \sum_{\lambda<0} \bar{a}_\lambda \chi(E_{-\lambda}) = 0.$$

But

$$\chi_0(f) = \overline{\chi_0(f^*)},$$

whence $\chi_0(f)=0$.

Take now any ϕ in $\mathfrak{A}'$. Then $\phi=\sum_{\lambda\in G}\phi_\lambda E_\lambda$, $\sum|\phi_\lambda|<\infty$. Since each E_λ with $\lambda>0$ is in $\mathfrak{A}'$, $\sum_{\lambda<0}\phi_\lambda E_\lambda$ is in $\mathfrak{A}'$. Fix $\lambda_0>0$. Then

$$\left(\sum_{\lambda<0}\phi_\lambda E_\lambda\right)E_{\lambda_0} = \sum_{\lambda<-\lambda_0}\phi_\lambda E_{\lambda+\lambda_0} + \sum_{\lambda\geqq-\lambda_0}\phi_\lambda E_{\lambda+\lambda_0}.$$

The left-hand term and the second term on the right are in $\mathfrak{A}'$. Hence $\sum_{\lambda<-\lambda_0}\phi_\lambda E_{\lambda+\lambda_0}$ is in $\mathfrak{A}'$. By the preceding, then,

$$0 = \chi_0\left(\sum_{\lambda<-\lambda_0}\phi_\lambda E_{\lambda+\lambda_0}\right) = \chi_0\left(\sum_{\lambda<-\lambda_0}\phi_\lambda E_\lambda\right)\cdot\chi_0(E_{\lambda_0}).$$

Hence $\chi_0(\sum_{\lambda<-\lambda_0}\phi_\lambda E_\lambda)=0$, since $|\chi_0(E_{\lambda_0})|=1$. Choose λ_1, $0\leqq\lambda_1<\lambda_0$. We then have $\chi_0(\sum_{\lambda<-\lambda_1}\phi_\lambda E_\lambda)=0$. Hence $\chi_0(\sum_{-\lambda_0\leqq\lambda<-\lambda_1}\phi_\lambda E_\lambda)=0$ and so $|\chi_0(\phi_{-\lambda_0}E_{-\lambda_0})|=|\chi_0(\sum_{-\lambda_0<\lambda<-\lambda_1}\phi_\lambda E_\lambda)|\leqq\|\sum_{-\lambda_0<\lambda<-\lambda_1}\phi_\lambda E_\lambda\|$. Since $\sum|\phi_\lambda|<\infty$, we can choose λ_1 so that $\|\sum_{-\lambda_0<\lambda<-\lambda_1}\phi_\lambda E_\lambda\| =\sum_{-\lambda_0<\lambda<-\lambda_1}|\phi_\lambda|<\epsilon$, for any given positive ϵ. Then $|\phi_{-\lambda_0}|=|\chi_0(\phi_{-\lambda_0}E_{-\lambda_0})|<\epsilon$. Hence $\phi_{-\lambda_0}=0$. Since this holds for all $\lambda_0>0$, we conclude that ϕ is in L^+. Hence $\mathfrak{A}'=L^+$, as asserted.

Note. I had first proved Theorem 1 for the group G of integers (Bull. Amer. Math. Soc. Abstract 60-2-281), but did not publish the proof. After hearing from R. Arens and I. M. Singer about their work on archimedean ordered groups [2] I tried to extend my result to that situation and proved Theorem 1 given in this paper.

References

1. G. W. Mackey, *Commutative Banach algebras*, Lecture notes (multigraphed), Harvard University, 1952, ed. by A. Blair, p. 95.

2. R. Arens and I. M. Singer, to appear.

Brown University