A \textit{W}-surface is a surface in ordinary Euclidean space for which there is a functional relation between the principal curvatures \( k_1, k_2 \):

\begin{equation}
W(k_1, k_2) = 0.
\end{equation}

We shall be interested in those \textit{W}-surfaces for which (1) can be written in the form

\begin{equation}
f(H, \mu) = 0, \quad \mu = H^2 - K,
\end{equation}

where \( H \) and \( K \) are the mean curvature and the Gaussian curvature respectively and where \( f \) is of class \( C^1 \). For such \textit{W}-surfaces we have

\begin{equation}
\frac{\partial W}{\partial k_1} = f_H/2 + (k_1 - k_2)f_\mu/2,
\end{equation}

\begin{equation}
\frac{\partial W}{\partial k_2} = f_H/2 - (k_1 - k_2)f_\mu/2.
\end{equation}

It follows that at an umbilic \((k_1 = k_2)\) we have

\begin{equation}
\frac{\partial W}{\partial k_1} \cdot \frac{\partial W}{\partial k_2} = f_H^2/4 \geq 0.
\end{equation}

A \textit{W}-surface (2) is called special if \( f_H \neq 0 \) at every umbilic.

Apparently there are very few closed special \textit{W}-surfaces.\(^1\) The following theorem is due to H. Hopf:\(^2\)

\textit{The only closed special analytic \textit{W}-surfaces of genus zero are spheres.}

It was P. Hartman and A. Wintner\(^3\) who succeeded in removing the analyticity assumption in Hopf's Theorem. Their theorem can be stated as follows:

\textit{Let \( S \) be a closed special \textit{W}-surface of genus zero, which is \( C^3 \)-imbedded in Euclidean space. Then \( S \) is a sphere.}

We shall show that the formalism developed in the preceding paper gives a very simple proof of the theorem of Hartman-Wintner.\(^4\)

\(^1\) The notion of a special \textit{W}-surface was first introduced by the author in the paper \textit{Some new characterizations of the Euclidean sphere}, Duke Math. J. vol. 12 (1945) pp. 279–290. I take this opportunity to mention that the proof of Theorem 3 in that paper is not valid. So far as I know, the question whether there exist closed surfaces of constant mean curvature and of genus \( > 0 \) remains unanswered.


\(^4\) The hypotheses of Hartman-Wintner are weakened in one respect in that the function \( f \) in (2) is here assumed to be of class \( C^1 \), while they supposed \( f \) to be of class \( C^2 \).
Let $x$, $y$ be isothermal parameters, and let $z = x + iy$. Let $E(dx^2 + dy^2)$, $Ldx^2 + 2Mdx dy + Ndy^2$ be respectively the first and second fundamental forms of the surface, so that
\begin{align}
2EH &= L + N, \\
E^2K &= LN - M^2.
\end{align}

Put
\begin{equation}
w = (L - N)/2 - iM.
\end{equation}

Then Codazzi’s equations can be written
\begin{equation}
w_\bar{z} = E_2 H_z.
\end{equation}

Also we have from (5)
\begin{equation}
\mu = H^2 - K = w\bar{w}/E^2.
\end{equation}

If the surface is a special $W$-surface satisfying (2), we have, in a neighborhood of an umbilic,
\begin{equation}
H_z = - (f_\mu/f_H)\mu_z.
\end{equation}

This means that $w$ satisfies a nonlinear differential equation of the form
\begin{equation}
w_\bar{z} = P(w\bar{w})_z + Qw\bar{w},
\end{equation}
where
\begin{equation}
P = - f_\mu/Ef_H, \quad Q = 2Ez/E^2 f_\mu/f_H.
\end{equation}

Following the procedure of Hartman-Wintner, the proof of their theorem depends on the lemmas:

**Lemma 1.** Let $w(z, \bar{z})$ be a solution of (9), in a sufficiently small neighborhood of $z = 0$, at which $w = 0$. Then $\lim_{z \to 0} w(z, \bar{z})z^{-k}$ exists if $w = o(\lvert z \rvert^{k-1})$.

**Lemma 2.** Under the hypotheses of Lemma 1, suppose that $w = o(\lvert z \rvert^{k-1})$ for all $k$. Then $w(z, \bar{z}) = 0$ in a neighborhood of $z = 0$.

From these lemmas we derive immediately the theorem of Hartman-Wintner. In fact, it follows from Lemma 2 that if $0$ ($z = 0$) does not have a neighborhood which consists entirely of umbilics, there exists an integer $k$, such that $w = o(\lvert z \rvert^{k-1})$, $w \not= o(\lvert z \rvert^k)$. By Lemma 1, $\lim_{z \to 0} w(z, \bar{z})z^{-k}$ exists and is $\not= 0$. We can therefore write
\begin{equation}
w(z, \bar{z}) = cz^k + o(\lvert z \rvert^k) \quad c \not= 0.
\end{equation}
It follows that the umbilic 0 is isolated and has an index \(-k<0\). By well-known arguments this implies the theorem of Hartman-Wintner.

It remains to prove the above lemmas. For this purpose let \(D\) be a disc of radius \(R\) about 0, and \(C\) its boundary circle. There exists a constant \(A>0\), such that in \(D\),

\[
|P(w\bar{w})_z + Qw\bar{w}| \leq A |w|.
\]

Suppose that \(w = o(|z|^{k-1})\). Let \(\zeta = \xi + i\eta\) be an interior point of \(D\). Then we have, for \(\zeta \neq 0\),

\[
\frac{d}{dz} \left\{ \frac{w dz}{z^k(z-\zeta)} \right\} = \frac{P(w\bar{w})_z + Qw\bar{w}}{z^k(z-\zeta)} d\bar{z} \wedge dz.
\]

Application of Stokes Theorem gives

\[
-2\pi i w(\zeta, \bar{\zeta})\zeta^{-k} + \int_C \frac{wdz}{z^k(z-\zeta)} = \int_D \int \frac{P(w\bar{w})_z + Qw\bar{w}}{z^k(z-\zeta)} d\bar{z}dz.
\]

It follows that

\[
2\pi |w(\zeta, \bar{\zeta})\zeta^{-k}| \leq \int_C \left| \frac{w(z, \bar{z})}{z^k(z-\zeta)} \right| dz
\]

\[
+ 2A \int_D \int \left| \frac{w(z, \bar{z})}{z^k(z-\zeta)} \right| dxdy.
\]

We multiply this inequality by \(d\zeta d\eta / |\zeta - \zeta_0|\), \(\zeta_0 \in D\), and integrate over \(D\). Remembering that

\[
\int_D \int \frac{dxdy}{|z-\zeta|} < 2R,
\]

\[
\frac{1}{|(z-\zeta)(\zeta_0-\zeta)|} = \frac{1}{|z-\zeta_0|} \left| \frac{1}{z-\zeta} + \frac{1}{\zeta-\zeta_0} \right|,
\]

we get from this integration

\[
2\pi \int_D \int \left| \frac{w(z, \bar{z})}{z^k(z-\zeta)} \right| dxdy \leq 4R \int_C \left| \frac{w(z, \bar{z})}{z^k(z-\zeta)} \right| dz
\]

\[
+ 8AR \int_D \int \left| \frac{w(z, \bar{z})}{z^k(z-\zeta)} \right| dxdy
\]

or

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We choose $R$ so small that $2\pi - 8AR > 0$. Then

$$
\int \int_D |w(z, \bar{z})/z^k(z - \zeta)| \, dx \, dy
$$

is bounded as $\zeta \to 0$, and the same is true of $|w(\zeta, \bar{\zeta})\zeta^{-k}|$. It follows that $|(w\bar{w})\zeta^{-k}|$ is bounded and from (14) that $\lim_{r \to 0} w(\zeta, \bar{\zeta})\zeta^{-k}$ exists. This proves Lemma 1.

To prove Lemma 2 we multiply (15) by $d\xi d\eta$ and integrate over $D$. This gives

$$
2\pi \int \int_D |w(z, \bar{z})z^{-k}| \, dx \, dy \leq 2R \int_C |w(z, \bar{z})z^{-k}| \, dz
$$

$$
+ 4AR \int \int_D |wz^{-k}| \, dx \, dy
$$

or

$$
(2\pi - 4AR) \int \int_D |w(z, \bar{z})z^{-k}| \, dx \, dy \leq 2R \int_C |w(z, \bar{z})z^{-k}| \, dz
$$

Suppose there exists a $z_0$ such that $w(z_0, \bar{z}_0) \neq 0$, $|z_0| < R$. Then the left-hand side of the above inequality is $\geq a|z_0|^{-k}$, and the right-hand side is $\leq bR^{-k}$, where $a$ and $b$ are positive constants independent of $k$. The hypothesis of Lemma 2 implies that $|z_0/R|^{-k} \geq a/b$ for all $k$, which is a contradiction. It follows that $w(z, \bar{z})$ vanishes identically for $|z| < R$. 

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