ON SPECIAL W-SURFACES

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A W-surface is a surface in ordinary Euclidean space for which there is a functional relation between the principal curvatures k_1, k_2:

$$W(k_1, k_2) = 0. \tag{1}$$

We shall be interested in those W-surfaces for which (1) can be written in the form

$$f(H, \mu) = 0, \qquad \mu = H^2 - K, \tag{2}$$

where H and K are the mean curvature and the Gaussian curvature respectively and where f is of class C^1. For such W-surfaces we have

$$\begin{aligned} \partial W/\partial k_1 &= f_H/2 + (k_1 - k_2)f_\mu/2, \\ \partial W/\partial k_2 &= f_H/2 - (k_1 - k_2)f_\mu/2. \end{aligned} \tag{3}$$

It follows that at an umbilic ($k_1 = k_2$) we have

$$\partial W/\partial k_1 \cdot \partial W/\partial k_2 = f_H^2/4 \geqq 0. \tag{4}$$

A W-surface (2) is called special if $f_H \neq 0$ at every umbilic.

Apparently there are very few closed special W-surfaces.[1] The following theorem is due to H. Hopf:[2]

The only closed special analytic W-surfaces of genus zero are spheres.

It was P. Hartman and A. Wintner[3] who succeeded in removing the analyticity assumption in Hopf's Theorem. Their theorem can be stated as follows:

Let S be a closed special W-surface of genus zero, which is C^3-imbedded in Euclidean space. Then S is a sphere.

We shall show that the formalism developed in the preceding paper gives a very simple proof of the theorem of Hartman-Wintner.[4]

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[1] The notion of a special W-surface was first introduced by the author in the paper *Some new characterizations of the Euclidean sphere*, Duke Math. J. vol. 12 (1945) pp. 279–290. I take this opportunity to mention that the proof of Theorem 3 in that paper is not valid. So far as I know, the question whether there exist closed surfaces of constant mean curvature and of genus >0 remains unanswered.

[2] H. Hopf, *Über Flächen mit einer Relation zwischen den Hauptkrümmungen*, Mathematische Nachrichten vol. 4 (1950–1951) pp. 232–249.

[3] P. Hartman and A. Wintner, *Umbilical points and W-surfaces*, Amer. J. Math. vol. 76 (1954) pp. 502–508.

[4] The hypotheses of Hartman-Wintner are weakened in one respect in that the function f in (2) is here assumed to be of class C^1, while they supposed f to be of class C^2.

Let x, y be isothermal parameters, and let $z=x+iy$. Let $E(dx^2+dy^2)$, $Ldx^2+2Mdxdy+Ndy^2$ be respectively the first and second fundamental forms of the surface, so that

$$2EH = L + N,\qquad E^2K = LN - M^2. \tag{5}$$

Put

$$w = (L - N)/2 - iM. \tag{6}$$

Then Codazzi's equations can be written

$$w_{\bar z} = EH_z. \tag{7}$$

Also we have from (5)

$$\mu = H^2 - K = w\bar{w}/E^2. \tag{8}$$

If the surface is a special W-surface satisfying (2), we have, in a neighborhood of an umbilic,

$$H_z = -(f_\mu/f_H)\mu_z.$$

This means that w satisfies a nonlinear differential equation of the form

$$w_{\bar z} = P(w\bar{w})_z + Qw\bar{w}, \tag{9}$$

where

$$P = -f_\mu/Ef_H,\qquad Q = 2E_z/E^2\cdot f_\mu/f_H. \tag{10}$$

Following the procedure of Hartman-Wintner, the proof of their theorem depends on the lemmas:

LEMMA 1. *Let* $w(z, \bar z)$ *be a solution of* (9), *in a sufficiently small neighborhood of* $z=0$, *at which* $w=0$. *Then* $\lim_{z\to 0} w(z, \bar z)z^{-k}$ *exists if* $w=o(|z|^{k-1})$.

LEMMA 2. *Under the hypotheses of Lemma* 1, *suppose that* $w=o(|z|^{k-1})$ *for all* k. *Then* $w(z, \bar z)\equiv 0$ *in a neighborhood of* $z=0$.

From these lemmas we derive immediately the theorem of Hartman-Wintner. In fact, it follows from Lemma 2 that if 0 ($z=0$) does not have a neighborhood which consists entirely of umbilics, there exists an integer k, such that $w=o(|z|^{k-1})$, $w\neq o(|z|^k)$. By Lemma 1, $\lim_{z\to 0} w(z, \bar z)z^{-k}$ exists and is $\neq 0$. We can therefore write

$$w(z, \bar z) = cz^k + o(|z|^k)\qquad c\neq 0. \tag{11}$$

It follows that the umbilic 0 is isolated and has an index $-k<0$. By well-known arguments this implies the theorem of Hartman-Wintner.

It remains to prove the above lemmas. For this purpose let D be a disc of radius R about 0, and C its boundary circle. There exists a constant $A>0$, such that in D,

$$|P(w\bar{w})_z + Qw\bar{w}| \leqq A|w|. \tag{12}$$

Suppose that $w=o(|z|^{k-1})$. Let $\zeta=\xi+i\eta$ be an interior point of D. Then we have, for $\zeta\neq 0$,

$$d\left\{\frac{wdz}{z^k(z-\zeta)}\right\} = \frac{P(w\bar{w})_z + Qw\bar{w}}{z^k(z-\zeta)} d\bar{z}\wedge dz. \tag{13}$$

Application of Stokes Theorem gives

$$-2\pi i w(\zeta,\bar{\zeta})\zeta^{-k} + \int_C \frac{wdz}{z^k(z-\zeta)} = \iint_D \frac{P(w\bar{w})_z + Qw\bar{w}}{z^k(z-\zeta)} d\bar{z}dz. \tag{14}$$

It follows that

$$\begin{aligned} 2\pi|w(\zeta,\bar{\zeta})\zeta^{-k}| \leqq& \int_C \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| |dz| \\ &+ 2A\iint_D \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| dxdy. \end{aligned} \tag{15}$$

We multiply this inequality by $d\xi d\eta/|\zeta-z_0|$, $z_0\in D$, and integrate over D. Remembering that

$$\iint_D \frac{dxdy}{|z-\zeta|} < 2R, \tag{16}$$

$$\frac{1}{|(z-\zeta)(z_0-\zeta)|} = \frac{1}{|z-z_0|}\left|\frac{1}{z-\zeta}+\frac{1}{\zeta-z_0}\right|, \tag{17}$$

we get from this integration

$$\begin{aligned} 2\pi\iint_D \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| dxdy \leqq& 4R\int_C \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| |dz| \\ &+8AR\iint_D \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| dxdy \end{aligned}$$

or

$$(18)\qquad (2\pi - 8AR)\iint_D \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| dxdy \leqq 4R\int_C \left|\frac{w(z,\bar{z})}{z^k(z-\zeta)}\right| |dz|.$$

We choose R so small that $2\pi - 8AR > 0$. Then

$$\iint_D |w(z,\bar{z})/z^k(z-\zeta)|\, dxdy$$

is bounded as $\zeta \to 0$, and the same is true of $|w(\zeta,\bar{\zeta})\zeta^{-k}|$. It follows that $|(w\bar{w})_\zeta \zeta^{-k}|$ is bounded and from (14) that $\lim_{\zeta\to 0} w(\zeta,\bar{\zeta})\zeta^{-k}$ exists. This proves Lemma 1.

To prove Lemma 2 we multiply (15) by $d\xi d\eta$ and integrate over D. This gives

$$2\pi \iint_D |w(z,\bar{z})z^{-k}|\, dxdy \leqq 2R\int_C |w(z,\bar{z})z^{-k}|\,|dz| + 4AR\iint_D |wz^{-k}|\, dxdy$$

or

$$(2\pi - 4AR)\iint_D |w(z,\bar{z})z^{-k}|\, dxdy \leqq 2R\int_C |w(z,\bar{z})z^{-k}|\,|dz|.$$

Suppose there exists a z_0 such that $w(z_0,\bar{z}_0) \neq 0$, $|z_0| < R$. Then the left-hand side of the above inequality is $\geqq a|z_0|^{-k}$, and the right-hand side is $\leqq bR^{-k}$, where a and b are positive constants independent of k. The hypothesis of Lemma 2 implies that $|z_0/R|^k \geqq a/b$ for all k, which is a contradiction. It follows that $w(z,\bar{z})$ vanishes identically for $|z| < R$.

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