COMPLETENESS, FULL COMPLETENESS, AND k SPACES

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The author in [2] defined full completeness for linear topological spaces as follows: if X is a real l.t.s. and if, for $U \subset X$, U^0 denotes the polar of U in the adjoint space X^* of continuous linear functionals on X, then X is said to be *fully complete* if a linear subspace L of X^* is weak* closed whenever $L \cap U^0$ is weak* closed, for every neighborhood U of zero in X. It was shown in [2, Corollary 14.1 and Theorem 16] and, independently, by Pták in [6, p. 350 and p. 336] that full completeness is stronger in general than completeness, and that every complete metrisable l.t.s. is fully complete. In addition, the author in [2, Corollary 17.2] proved that an arbitrary cartesian product of reals (with the product topology) is fully complete, and in [6, p. 330] it was shown that full completeness is equivalent (for locally convex X) to the following: every continuous linear function on X onto another locally convex l.t.s. which takes open sets into somewhere dense sets is already open (actually, this last property was Pták's main concern and was labeled B-completeness by him).

Our purpose here is to examine this concept in $X = C(E)$, where E is a completely regular T_1 topological space and $C(E)$ is the l.t.s. of real-valued continuous functions on E, with the compact-open topology. More precisely, we study the relations between completeness and full completeness for $C(E)$ and certain related concepts in E for two particular classes of spaces E: pseudo-finite E and hemicompact E. By definition, the space E is *pseudo-finite* if every compact set of E is finite (the P-spaces recently considered in [4] furnish examples of pseudo-finite spaces, and as is pointed out there, many nondiscrete P-spaces exist), and E is *hemicompact* [see 1] if there exists a countable family $\mathcal{K}$ of compact subsets of E whose union is E and such that each compact set is contained in some member of $\mathcal{K}$. Two final definitions needed are those of a k space (see [3]) and of the k extension of a topology. The *k extension* of the given topology of E is the strongest topology on E which agrees with the given topology on each compact set (we denote this derived topology henceforth by k), and E is a *k space* if the topology k coincides with the original topology. Pták proved in [6, pp. 342–343] that E is a k space when $C(E)$ is fully complete, and gave an example [6, p. 350] to show the

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converse need not be true. Implicit in his paper is the fact that E is normal when $C(E)$ is fully complete.

Our first results below include a characterization of k spaces (for completely regular spaces) together with a brief discussion of pseudo-finite spaces and hemicompact spaces. It turns out that one of the conditions in this characterization is that E endowed with k be completely regular; in this case we say simply that *k is completely regular.* It is emphasized here that the original space E is assumed to be completely regular throughout this paper.

LEMMA 1. *The following two conditions together are necessary and sufficient that E be a k space:* (1) *$C(E)$ is complete, and* (2) *k is completely regular.*

PROOF. We point out that the requirement that $C(E)$ be complete is equivalent to the statement that every real-valued function on E whose restriction to each compact set K is continuous in the relative topology of K is already continuous.

Necessity. Since E is a k space and thus k coincides with the original topology, assumed to be completely regular, it is clear that k is completely regular. Further, it is known [**5**, p. 76] that every k space E has $C(E)$ complete.

Sufficiency. Denote by $A(E)$ the family of real-valued functions on E whose restrictions to each compact set are continuous and by T_A the weakest topology on E for which all the members of $A(E)$ remain continuous. It is clear that each set open with respect to the original topology T is T_A open and that each T_A open set is k open. If $C(E)$ is complete then $A(E)=C(E)$; hence T and T_A coincide, since T is completely regular. If k is completely regular then k and T_A coincide; but then E is a k space.

LEMMA 2. *If E is either pseudo-finite or hemicompact then the topology k is completely regular.*

PROOF. Lemma 3 below together with the fact that every discrete space is completely regular make clear the statement for E pseudo-finite.

The author in [**2**, Theorem 12] proved that if X is an l.t.s., if T denotes the strongest topology for X^* which agrees with the weak* topology on each U^0 (U a neighborhood of zero in X), and if t denotes the topology for X^* of uniform convergence on totally bounded sets of X, then X metrisable implies T and t are the same. Now, let M be the image of E under the evaluation map e, where, for $x\in E$, $e_x(f)=f(x)$, all $f\in C(E)$. Then $M\subset X^*$, where $X=C(E)$, and it is easily shown that e is a homeomorphism between E and M when E is given the topology k and M is given its relative T topology. By our initial remarks, since E hemicompact implies X is metrisable, T and t coincide; in particular they coincide on M. But then k is completely regular, since t is completely regular. This concludes the proof.

The permanence properties of pseudo-finite spaces are easily established. For example, every subspace of a pseudo-finite space is pseudo-finite, every space which is a pairwise disjoint union of open and closed pseudo-finite subspaces is pseudo-finite, every compact pseudo-finite space is finite, and every finite product of pseudo-finite spaces is pseudo-finite (an infinite product need not be). In addition, we have

LEMMA 3. *The following conditions are equivalent for* E: (1) *E is pseudo-finite*, (2) *the compact-open and point-open topologies for* $C(E)$ *coincide, and* (3) *k coincides with the discrete topology.*

PROOF. (1) implies (2) is clear, since each compact set is finite when E is pseudo-finite. To see that (2) implies (1), let M be compact. Then, $M^0=[f: f\in C(E)$ and $|f(t)|\leqq 1$, all $t\in M]$ contains a neighborhood of zero in $C(E)$; hence, by (2), there exists N finite in E such that $N^0\subset M^0$. But then $(M^0)_0\subset(N^0)_0$, where for A in $C(E)$, $A_0=[t: t\in E$ and $|f(t)|\leqq 1$, all $f\in A]$. However, E completely regular implies (since M and N are closed) that $(M^0)_0=M$, $(N^0)_0=N$; thus $M\subset N$, M is finite, and E is pseudo-finite. Now assume (1) and let $F\subset E$ be any set. If M is compact, then $F\cap M$ is either void or finite (since M is finite), and thus $F\cap M$ is closed. Therefore, by definition of k, F is k closed; i.e., every set is k closed and k is the discrete topology. Conversely, if k is the discrete topology of E, then every compact set in E is finite, since this is true of the discrete topology and since the topology k and the original topology have the same compact sets.

THEOREM 1. *For E a pseudo-finite space the following conditions are equivalent*: (1) *E is discrete*, (2) *E is locally compact*, (3) *$C(E)$ is complete*, (4) *$C(E)=R^E$, where R^E is the cartesian product of $|E|$ copies of the reals, with the product topology*, (5) *E is a k space, and* (6) *$C(E)$ is fully complete.*

PROOF. The implication $(1)\cdot\rightarrow\cdot(2)$ is clear, since every discrete space is locally compact, and $(2)\cdot\rightarrow\cdot(3)$ follows from the known fact that $C(E)$ is complete when E is locally compact. If (3) holds, then every real function on E is continuous, since every function has its restriction to each finite set continuous; i.e., $C(E)=R^E$ (as sets). To complete $(3)\cdot\rightarrow\cdot(4)$, note that the product topology on R^E is the compact-open topology when E is pseudo-finite. The implication (4)$\cdot\rightarrow\cdot$(6) follows from the remark of the first paragraph of this paper, and (6)$\cdot\rightarrow\cdot$(5) was shown by Pták in [6, pp. 342–343] for arbitrary completely regular E, and can be derived directly here by making use of completeness and Lemmas 2 and 1. Finally, (5)$\cdot\rightarrow\cdot$(1) follows from Lemma 3. This completes the proof.

THEOREM 2. *For E a hemicompact space the following conditions are equivalent*: (1) $C(E)$ *is complete*, (2) E *is a k space, and* (3) $C(E)$ *is fully complete.*

PROOF. The implication (1)$\cdot\rightarrow\cdot$(2) follows from Lemmas 1 and 2, and (2)$\cdot\rightarrow\cdot$(3)$\cdot\rightarrow\cdot$(1) have already been indicated in the remarks preceding this theorem (the fact that $C(E)$ is metrisable here is needed in (2)$\cdot\rightarrow\cdot$(3)).

Many questions remain unanswered concerning the relations between completeness and full completeness for $C(E)$ and the topology k in E. Whether the conclusion of Theorem 2 holds for spaces other than hemicompact or pseudo-finite E we do not know. As we pointed out in paragraph 2, $C(E)$ fully complete implies E is normal, and this fact together with the fact that both hemicompact E and discrete E are paracompact suggests the possibility that E is paracompact when $C(E)$ is fully complete. It would be interesting to find necessary and sufficient conditions on E that $C(E)$ be fully complete. Finally, the conclusion of Lemma 2 holds also for locally compact E and for spaces E satisfying the first axiom of countability, since these spaces are known to be k spaces [3].

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