AN ADDITION THEOREM FOR LEBESGUE AREA[1]

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Let X be the space of a finite 2-dimensional cell complex K, in the sense that K is a finite set of compact topological cells of dimensions ≤ 2 with disjoint interiors, with union X, and such that the boundary of each element of K is the union of certain lower dimensional elements of K. Also let f be a continuous map of X into Euclidean n-space, with $n \geq 2$. Observing that X is finitely triangulable and designating k-dimensional Lebesgue area by L_k, we may consider the area $L_2(f)$, the area $L_2(f|A)$ of the restriction of f to each 2-cell A in K, and the length $L_1(f|B)$ of the restriction of f to each 1-cell B in K. We shall prove the following additivity theorem:

If $L_1(f|B) < \infty$ for each 1-cell B in K, then

$$L_2(f) = \sum_A L_2(f|A),$$

where the summation extends over all 2-cells A in K.

This result is the solution of a classical problem raised by the original work of Lebesgue [1]. It provides one of the basic tools for the extension of the whole theory of Lebesgue area to continuous maps of a 2-manifold into n-space, which is presently being carried out by Paul Slepian.

For the special case in which X is a 2-cell the additivity theorem follows immediately (see [2, page 506] for $n \leq 3$) from the known equality between the Lebesgue area of f and any one of several areas which may be defined in terms of the Lebesgue areas of the 2-dimensional projections of f. In fact it is obvious that the addition formula holds with L_2 replaced by the 2-dimensional integralgeometric stable area (see [3, page 319]), which agrees (according to [4, 7.16]), with L_2 for maps of a 2-cell into E_n.

Next consider the special case in which f maps each 1-cell in K onto the union of finitely many straight line segments in n-space. Then the f image of the 1-skeleton of K is 2-removable (by [4, 6.26, 6.27, 6.30]), hence $L_2(f)$ is the supremum of $L_2(f|Y)$ over all those finitely triangulable subsets Y of X which do not meet the 1-skeleton of K. Now

$$L_2(f \mid Y) = \sum_A L_2(f \mid Y \cap A)$$

for any such set Y, because the intersections of Y with the 1-cells in K are disjoint. Hence passage to the supremum yields the addition formula for the present special case.

The general additivity theorem may be derived from the preceding two special cases as follows:

Suppose $\epsilon > 0$.

For each 1-cell B in K the curve $f \mid B$ is rectifiable, by hypothesis, and we can therefore select[2] a homeomorphism h_B which maps the interval $\{t \mid 0 \le t \le 1\}$ onto B in such a way that

$$\left| (f \circ h_B)(t_2) - (f \circ h_B)(t_1) \right| \le [L_1(f \mid B) + \epsilon] \cdot (t_2 - t_1)$$

whenever $0 \le t_1 \le t_2 \le 1$.

For each 2-cell A in K we let $B_1(A), B_2(A), \cdots, B_{\mu(A)}(A)$ be those 1-cells in K which are contained in the boundary of A, ordered in such a way that we can select a homeomorphism h_A of the circular disc $\{z \mid |z| \le 1\}$ onto A for which $h_A(e^{i\theta}) = h_{B_{j(A)}}[\mu(A)\theta/2\pi - j + 1]$ whenever $(j-1)/\mu(A) \le \theta/2\pi \le j/\mu(A)$ and $j = 1, 2, \cdots, \mu(A)$.

Associating with each 1-cell B in K the number $\nu(B)$ of 2-cells A in K such that B is contained in the boundary of A, we choose an integer

$$q > \epsilon^{-1} \sum_B \nu(B)[L_1(f \mid B) + \epsilon]^2,$$

where the summation extends over all 1-cells B in K, and construct a continuous map g of X into n-space which agrees with f at the vertices of K and which is characterized by the following two properties:

If B is a 1-cell in K, then $g \circ h_B$ is the polygonal function with $q+1$ vertices which agrees with $f \circ h_B$ at $0, 1/q, 2/q, \cdots, 1$.

If A is a 2-cell in K, then

$$(g \circ h_A)(re^{i\theta}) = (1+q)(1-r)(f \circ h_A)(e^{i\theta}) + [r(q+1) - q](g \circ h_A)(e^{i\theta})$$

in case $q/(q+1) \le r \le 1$, and

$$(g \circ h_A)(re^{i\theta}) = (f \circ h_A)\left(\frac{q+1}{q} re^{i\theta}\right)$$

in case $0 \le r \le q/(q+1)$.

Since g maps each 1-cell in K onto the union of finitely many

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[2] In case $B = \{t \mid 0 \le t \le 1\}$, let h_B be the inverse of the function which maps t onto $(L_1(f \mid \{u \mid 0 \le u \le t\}) + \epsilon t)/(L_1(f \mid B) + \epsilon)$.

straight line segments in n-space, the addition formula holds with f replaced by g, so that

$$L_2(g) = \sum_A L_2(g \mid A).$$

Moreover for each 2-cell A in K we may apply the addition formula with f, X, and K replaced by $g \circ h_A$, $\{z \mid |z| \leq 1\}$, and a complex whose 2-cells are the circular disc

$$D = \left\{z \mid |z| \leq \frac{q}{q+1}\right\}$$

and the annular sectors

$$S_j = \left\{re^{i\theta} \mid \frac{q}{q+1} \leq r \leq 1 \text{ and } \frac{j-1}{\mu(A)} \leq \frac{\theta}{2\pi} \leq \frac{j}{\mu(A)}\right\}$$

corresponding to $j = 1, 2, \cdots, \mu(A)$; we find that

$$L_2(g \mid A) = L_2(g \circ h_A) = L_2(g \circ h_A \mid D) + \sum_{j=1}^{\mu(A)} L_2(g \circ h_A \mid S_j).$$

Evidently

$$L_2(g \circ h_A \mid D) = L_2(f \circ h_A) = L_2(f \mid A).$$

For each j we map the square

$$Q = \{(s, t) \mid 0 \leq s \leq 1 \text{ and } 0 \leq t \leq 1\}$$

homeomorphically onto S_j by mapping (s, t) onto

$$\frac{q+s}{q+1} \exp\left[\frac{i2\pi}{\mu(A)}(t + j - 1)\right],$$

thus transforming $g \circ h_A \mid S_j$ into a function on Q which maps (s, t) onto

$$(1 - s)[f \circ h_{B_j(A)}](t) + s[g \circ h_{B_j(A)}](t).$$

At almost all points (s, t) in Q the two partial derivatives of this Lipschitzian function are the vectors

$$[g \circ h_{B_j(A)}](t) - [f \circ h_{B_j(A)}](t)$$

and

$$(1 - s)[f \circ h_{B_j(A)}]'(t) + s[g \circ h_{B_j(A)}]'(t),$$

whose norms are less than or equal to

$$q^{-1}(L_1[f \mid B_j(A)] + \epsilon),$$

$$L_1[f \mid B_j(A)] + \epsilon$$

respectively. Therefore the Jacobian integral over Q, which equals $L_2(g \circ h_A \mid S_j)$, does not exceed

$$q^{-1}(L_1[f \mid B_j(A)] + \epsilon)^2.$$

It follows that

$$L_2(g \mid A) \leqq L_2(f \mid A) + q^{-1} \sum_{j=1}^{\mu(A)} (L_1[f \mid B_j(A)] + \epsilon)^2.$$

Summing over all 2-cells A in K we obtain the inequality

$$L_2(g) \leqq \sum_A L_2(f \mid A) + \epsilon.$$

Inasmuch as g is (uniformly) close to f for large q, and in view of the arbitrary nature of ϵ, we infer that

$$L_2(f) \leqq \sum_A L_2(f \mid A).$$

We complete the proof by observing that the opposite inequality is an obvious consequence of [4, 6.25] and of the fact that, for each 2-cell A in K, $L_2(f \mid A)$ is the supremum of $L_2(f \mid C)$ over all 2-cells C contained in the interior of A.

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