A NOTE ON ESTIMATING DISTRIBUTION FUNCTIONS

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1. Statement of the problem. Let $A$ be a positive number and for each positive integer $n$ let $f_n(y)$ be a continuous function on the closed interval $[-A, A]$. Let $F(y)$ be a distribution function on $[-A, A]$. For each $n = 1, 2, \cdots$, define $a_n$ by

\begin{equation}
    a_n = \int_{-A}^{A} f_n(y) dF(y).
\end{equation}

In this note we consider the problem of estimating the distribution function $F(y)$ in terms of the sequence of numbers $\{a_n\}$, and the sequence of functions $f_n(y)$. To this end we consider, for each positive integer $n$, a system of equations and inequalities. We construct a distribution function $F_n(y)$ in terms of any solution of this system, and show that $\lim_{n \to \infty} F_n(y) = F(y)$ for every continuity point of $F(y)$.

2. Conditions for uniqueness of $F$. It is clear that in order to be able to estimate $F$, we must assume that $F$ is the unique distribution function satisfying (1.1). More precisely we shall make the following assumption.

Assumption. Let $G(y)$ be any function of bounded variation defined on $[-A, A]$ and satisfying

\begin{equation}
    a_n = \int_{-A}^{A} f_n(y) dG(y), \quad n = 1, 2, \cdots.
\end{equation}

Then $F(y) - G(y)$ is identically constant.

In this section we shall derive a condition which is equivalent to the uniqueness assumption. To this end let $B$ be the Banach space of continuous functions defined on $[-A, A]$ and normed by

\begin{equation}
    \|f\| = \max_{y \in [-A,A]} |f(y)|.
\end{equation}

Then we have

Theorem 1. A necessary and sufficient condition that $F$ be unique is that the sequence $f_n(y)$ be fundamental in $B$.

Proof. Suppose that $F$ is unique. Let $B'$ be the closed linear manifold spanned by the sequence $f_n$, and suppose that $B'$ is a proper
subspace of $B$. Let $f_0 \in B - B'$, and let $\phi$ be a bounded linear functional defined on $B$ with $\phi(f_0) = 1$, and $\phi(f) = 0$, for $f \in B'$. It is well known that such functionals exist. From the representation theorem for linear functionals on $B$ it follows that there exists a function of bounded variation on $[-A, A]$, say $H(y)$, satisfying

\begin{equation}
\phi(f) = \int_{-A}^{A} f(y) dH(y), \quad \text{for every } f \in B.
\end{equation}

Now let $G(y) = F(y) + H(y)$. Clearly $H(y)$ is not identically constant, for $\int_{-A}^{A} f_0(y) dH(y) = 1$. On the other hand we have

\begin{equation}
a_n = \int_{-A}^{A} f_n(y) dG(y), \quad n = 1, 2, \ldots,
\end{equation}

since $\int_{-A}^{A} f_n(y) dH(y) = 0$ for every $n$. Since $F$ is assumed to be unique, it follows that the sequence $f_n$ is fundamental, thus proving necessity.

Conversely suppose that the sequence $F_n$ is fundamental in $B$, and suppose that $G(y)$ is a function of bounded variation on $[-A, A]$ satisfying

\begin{equation}
\int_{-A}^{A} f_n(y) dF(y) = \int_{-A}^{A} f_n(y) dG(y), \quad n = 1, 2, \ldots.
\end{equation}

From the fact that strong convergence in $B$ implies weak convergence in $B$, and from the fact that the sequence $f_n$ is fundamental in $B$, it follows that equation (2.5) holds for every $f \in B$. Hence for every real number $t$ we have

\begin{equation}
\int_{-A}^{A} e^{itv} dF(y) = \int_{-A}^{A} e^{itv} dG(y),
\end{equation}

and the uniqueness of $F$ follows from well-known properties of Fourier-Stieltjes transforms.

3. Construction of the sequence $F_n$. Let $n$ be a given positive integer. Let $y_0 = -A, y_1, y_2, \ldots, y_n = A$ be a subdivision of $[-A, A]$ into $n$ equal subintervals. For $1 \leq i \leq n, 1 \leq j \leq n$, define the numbers $M_{ij}^{(n)}$ and $m_{ij}^{(n)}$ by

\begin{equation}
M_{ij}^{(n)} = \max_{v_{i-1} \leq v \leq v_j} f_i(y), \quad m_{ij}^{(n)} = \min_{v_{i-1} \leq v \leq v_j} f_i(y).
\end{equation}

Consider the following system of equations and inequalities in the unknowns $H_1^{(n)}, \ldots, H_n^{(n)}$:...
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(i) $H_j^{(n)} \geq 0, \quad j = 1, \ldots, n.$

(ii) $\sum_{j=1}^{n} H_j^{(n)} = 1.$

\begin{equation}
(3.2)
\end{equation}

(iii) $\sum_{j=1}^{n} M_{ij}^{(n)} H_j^{(n)} \geq a_i, \quad i = 1, \ldots, n.$

(iv) $\sum_{j=1}^{n} m_{ij}^{(n)} H_j^{(n)} \leq a_i, \quad i = 1, \ldots, n.$

The system (3.2) clearly has the solution $H_j^{(n)} = \int_{y_{j-1}}^{y_j} dF, \quad j = 1, \ldots, n.$

Now let $H_1^{(n)}, \ldots, H_n^{(n)}$ be an arbitrary solution of (3.2). We define a distribution function $F_n(y)$ on $[-A, A]$ by

\begin{equation}
(3.3)
F_n(y) = \sum_{y_{j-1} \leq y} H_j^{(n)}.
\end{equation}

In the next section we shall prove

**Theorem 2.** For each point of continuity of $F(y)$ we have

$$\lim_{n \to \infty} F_n(y) = F(y).$$

4. Proof of Theorem 2.

**Lemma 1.** Let $r$ be a fixed positive integer. Then $\lim_{n \to \infty} \int_{-A}^{A} A f_r(y) dF_n(y) = a_r.$

**Proof.** We have $\int_{-A}^{A} A f_r(y) dF_n(y) = \sum_{j=1}^{n} f_r(y_j) H_j^{(n)}$ for every positive integer $n$. Hence

$$\sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} \leq \int_{-A}^{A} f_r(y) dF_n(y) \leq \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)},$$

and it is sufficient to show that

$$a_r = \lim_{n \to \infty} \sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} = \lim_{n \to \infty} \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)}.$$

Now for each $n \geq r$, we have, in virtue of (3.2),

$$\sum_{j=1}^{n} m_{rj}^{(n)} H_j^{(n)} \leq a_r \leq \sum_{j=1}^{n} M_{rj}^{(n)} H_j^{(n)}.$$

Also

$$\sum_{j=1}^{n} [M_{rj}^{(n)} - m_{rj}^{(n)}] H_j^{(n)} \leq \max_{j=1, \ldots, n} [M_{rj}^{(n)} - m_{rj}^{(n)}].$$
Since \( f_r(y) \) is uniformly continuous on \([-A, A]\), the desired result follows.

**Lemma 2.** For each \( f \in B \), we have

\[
\lim_{n \to \infty} \int_{-A}^{A} f(y) dF_n(y) = \int_{-A}^{A} f(y) dF(y).
\]

**Proof.** Let \( f \in B \), and let \( \epsilon \) be a positive number. The sequence \( f_n \) is fundamental in \( B \), and so we may choose a finite subset, say \( f_{i_1}, \ldots, f_{i_r} \), and real numbers \( c_1, \ldots, c_r \) with the property that

\[
\left\| f - \sum_{j=1}^{r} c_j f_{i_j} \right\| < \epsilon/3.
\]

Without loss of generality we may assume that \( \sum_{i=1}^{r} |c_i| > 0 \). Now, from Lemma 1, we may choose an integer \( N \), so that for \( n \geq N \) we have

\[
\max_{j=1, \ldots, r} \left| \int_{-A}^{A} f_{i_j}(y) dF_n(y) - \int_{-A}^{A} f_{i_j}(y) dF(y) \right| < \epsilon/3 \sum_{j=1}^{r} |c_j|
\]

and consequently

\[
\left| \int_{-A}^{A} \left( \sum_{j=1}^{r} c_j f_{i_j}(y) \right) dF_n(y) - \int_{-A}^{A} \left( \sum_{j=1}^{r} c_j f_{i_j}(y) \right) dF(y) \right| < \frac{\epsilon}{3}.
\]

Then for \( n \geq N \), we have

\[
\left| \int_{-A}^{A} f(y) dF_n(y) - \int_{-A}^{A} f(y) dF(y) \right|
\]

\[
\leq \left| \int_{-A}^{A} f(y) dF_n(y) - \int_{-A}^{A} \left( \sum_{j=1}^{r} c_j f_{i_j}(y) \right) dF_n(y) \right|
\]

\[
+ \left| \int_{-A}^{A} \left( \sum_{j=1}^{r} c_j f_{i_j}(y) \right) dF_n(y) - \int_{-A}^{A} \left( \sum_{j=1}^{r} c_j f_{i_j}(y) \right) dF(y) \right|
\]

\[
+ \left| \int_{-A}^{A} f(y) dF(y) - \int_{-A}^{A} f(y) dF(y) \right|.
\]

The first and last terms are bounded by \( \| f - \sum_{j=1}^{r} c_j f_{i_j} \| \), and the middle term by \( \epsilon/3 \). Hence

\[
\left| \int_{-A}^{A} f(y) dF_n(y) - \int_{-A}^{A} f(y) dF(y) \right| < \epsilon.
\]
The proof of the theorem is now immediate. For each real number \( t \), let \( \psi_n(t) = \int_{A} e^{itv}dF_n(y) \), and let \( \psi(t) = \int_{A} e^{itv}dF(y) \). Then we have \( \lim_{n \to \infty} \psi_n(t) = \psi(t) \) for every \( t \), and the theorem follows from the continuity theorem for Fourier-Stieltjes transforms.

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**REAL-VALUED MAPPINGS OF SPHERES**

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This note concerns subsets \( \Delta \) of the unit 2-sphere \( S \) such that (*) for each continuous real-valued mapping \( f \) of \( S \) there exists a rotation \( r \) of \( S \) with all points of \( r(\Delta) \) having the same value under \( f \). In 1942, Kakutani [3] proved that the set \( \Delta \) of end points of an orthonormal set of 3 vectors has property (*). It was observed by de Mira Fernandes [5] that the same proof holds in case \( \Delta \) is the set of vertices of any equilateral triangle. Yamabe and Yuyobo [8] proved a generalization of Kakutani's theorem to \( n \)-space. Their method may be used to prove that the set \( \Delta \) of vertices of an isosceles triangle has property (*) (this has been carried out in a Master's thesis of R. D. Johnson [2]). Here we prove that the set \( \Delta \) of vertices of any triangle has property (*); the methods differ from both those of Kakutani and those of Yamabe and Yuyobo.

Dyson [1] has proved that the set of vertices of a square centered at the origin has property (*); Livesay [4] has extended this to any rectangle centered at the origin. The problem of finding all such sets \( \Delta \) having property (*) is unsolved.

**Theorem.** Let \( f \) be a continuous real-valued mapping of the sphere \( S \) and let \( x_0, x_1, x_2 \in S \). There exists a rotation \( r \) with \( f(r(x_0)) = f(r(x_1)) = f(r(x_2)) \).

We need the following lemma.

**Lemma.** Suppose that \( X \) is a unicoherent locally connected continuum, and that \( T \) is a map of period 2 on \( X \) without fixed points. Suppose \( A \) is a subset of \( X \) which (i) is closed in \( X \), (ii) is invariant under \( T \), and

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