1. The differential equation

\[ x'' + f(t)x = 0, \]

in which \( f = f(t) \) is a continuous function on the half-line \( 0 \leq t < \infty \), is said to be oscillatory if some (hence every) nontrivial solution \( x = x(t) \) possesses an infinity of zeros on \( 0 \leq t < \infty \) (clustering only at \( +\infty \)). Various criteria for the oscillatory nature of (1) are known; see, e.g., [1] and the references cited there.

It was shown by Wintner [2] that if \( F(t) \to \infty \) as \( t \to \infty \), where

\[ F(t) = \int_0^t f(s) ds, \]

or even if

\[ G(t) = \int_0^t F(s) ds \to \infty, \quad t \to \infty, \]

then (1) must be oscillatory. Various refinements as well as variations of the criterion (3) were obtained by Hartman in [1]. The present note will be devoted to the derivation of two further criteria, given in (*) and (**) below, involving the function \( G(t) \) of (3).

Let \( E(M, T) \) denote the set of points \( t \) of the half-line \( T \leq t < \infty \) for which the function \( G(t) \) of (3) satisfies the inequality \( G(t) > M \), where \( M \) is a positive constant. The following will be proved:

(*) Suppose that there exists a pair of sequences \( T_n, M_n \) satisfying \( T_n \to \infty, M_n \to \infty \) as \( n \to \infty \), and for which

\[ \exp (M_n T_n) \text{ meas } E(M_n, T_n) \to \infty, \quad n \to \infty. \]

Then the equation (1) is oscillatory.

Since (3) implies \( \text{meas } E(M, T) = \infty \) whenever \( M, T > 0 \), the sufficiency of (3) for the oscillatory nature of (1) is a consequence of (*). In fact, the proof of (*) will depend upon a refinement of the argument used by Wintner in [2] in obtaining the criterion (3).

It is known that if (3) is relaxed to

Received by the editors February 23, 1955.

1 This work was supported by the National Science Foundation research grant NSF-G481.
\[(5) \quad \limsup T^{-1} \int_0^T F(t) \, dt = \infty, \quad T \to \infty,\]

then \((1)\) need not be oscillatory; (see (II bis) of [1, p. 390]). It will be shown however, as a corollary of \((*)\), that this situation cannot occur if, for instance, \(f(t)\), or even \(F(t)\), is bounded exponentially from below. Thus,

\[(**) \quad \text{If, in addition to (5), the function } F(t) \text{ of (2) also satisfies} \]

\[(6) \quad F(t) > - \exp (Ct), \]

for some positive constant \(C\), then the equation \((1)\) is oscillatory.

2. Proof of \((*)\). If \(x = x(t)\) and \(y = y(t)\) denote two linearly independent solutions of \((1)\), it is clear from [2] that the equation \((1)\) is oscillatory if and only if

\[(7) \quad \int_0^\infty (x^2 + y^2)^{-1} \, dt = \infty. \]

In order to prove \((*)\), suppose, if possible, that \((1)\) is nonoscillatory. It will be shown that this assumption implies \((7)\), hence a contradiction, and the proof of \((*)\) will be complete. Since, for large values of \(t\), the logarithmic derivative, \(z\), of a solution of \((1)\) satisfies the Riccati equation \(z' + z^2 + f = 0\), the inequality \(x^2 + y^2 \leq \text{const.} \exp (-2\int_0^t F(s) \, ds + Kt),\) in which \(K\) denotes a constant, holds for \(0 \leq t < \infty\); cf. formula line (7) of [2]. Consequently,

\[(8) \quad \int_0^\infty (x^2 + y^2)^{-1} \, dt \geq \text{const.} \int_0^\infty \exp \left[ (2G - K) t \right] dt, \]

where \(G = G(t)\) is defined by (3). Let \(M > K\). In view of (8) and the inequalities

\[ \int_0^\infty \exp \left[ (2G - K) t \right] dt \geq \int_0^T \exp \left[ (2G - K) t \right] dt \geq \exp \left[ (2M - K) T \right] \text{meas } E(M, T), \]

it follows that

\[(9) \quad \int_0^\infty (x^2 + y^2)^{-1} \, dt \geq \text{const.} \exp (MT) \text{meas } E(M, T). \]

Since the left side of the inequality \((9)\) is independent of \(M\) and \(T\), relation \((4)\) implies \((7)\). This completes the proof of \((*)\).
3. **Proof of (**)**. In view of (6) and the relation \((tG)' = F\), the inequality
\[
(tG)' > - \exp(Ct)
\]
holds for some positive constant \(C\). If \(a \leq t \leq b\) and \(G(a) > 0\), a quadrature of (10) leads to \(tG(t) - aG(a) > - \int_a^t \exp(Cs)ds > -(b-a) \cdot \exp(Cb)\), and hence
\[
G(t) > ab^{-1}G(a) - a^{-1}(b-a) \exp(Cb), \quad a \leq t \leq b.
\]
According to (5), there exists a sequence \(t = t_1 < t_2 < \cdots\) such that \(t_n \to \infty\) and \(G(t_n) \to \infty\) as \(n \to \infty\). For a given \(M > 0\), choose \(a = t_n\) (for some \(n\) depending on \(M\)) such that \(G(t_n) > 2M\), and let \(b\) be defined by \(b - a = \exp(-Cb)\). Then relation (11) implies \(G(t) > 2ab^{-1}M - a^{-1}\); hence, since \(b - a \to 0\) as \(a \to \infty\), \(G(t) > M\) for \(a \leq t \leq b\) and \(a\) sufficiently large. Consequently, the inequality
\[
\exp(Ma) \text{ meas } E(M, a) \geq \exp(Ma - Cb)
\]
holds for certain arbitrarily large numbers \(a\) and \(b = a + \exp(-Cb)\). Clearly, for every fixed \(M > C\), \(\exp(MT) \text{ meas } E(M, T) \to \infty\) for a sequence of \(T(=a)\) values tending to \(\infty\). In particular, relation (4) holds and (\(\ast\)) now implies (\(\ast\)).

**References**
