NOTE ON SOME OSCILLATION CRITERIA[1]

C. R. PUTNAM

1. The differential equation

$$x'' + fx = 0, \tag{1}$$

in which $f=f(t)$ is a continuous function on the half-line $0 \leqq t < \infty$, is said to be oscillatory if some (hence every) nontrivial solution $x=x(t)$ possesses an infinity of zeros on $0 \leqq t < \infty$ (clustering only at $+\infty$). Various criteria for the oscillatory nature of (1) are known; see, e.g., [1] and the references cited there.

It was shown by Wintner [2] that if $F(t) \to \infty$ as $t \to \infty$, where

$$F(t) = \int_0^t f(s)ds, \tag{2}$$

or even if

$$G(t) \equiv t^{-1} \int_0^t F(s)ds \to \infty, \qquad t \to \infty, \tag{3}$$

then (1) must be oscillatory. Various refinements as well as variations of the criterion (3) were obtained by Hartman in [1]. The present note will be devoted to the derivation of two further criteria, given in (*) and (**) below, involving the function $G(t)$ of (3).

Let $E(M, T)$ denote the set of points t of the half-line $T \leqq t < \infty$ for which the function $G(t)$ of (3) satisfies the inequality $G(t) > M$, where M is a positive constant. The following will be proved:

(*) *Suppose that there exists a pair of sequences T_n, M_n satisfying $T_n \to \infty$, $M_n \to \infty$ as $n \to \infty$, and for which*

$$\exp(M_n T_n) \text{ meas } E(M_n, T_n) \to \infty, \qquad n \to \infty. \tag{4}$$

Then the equation (1) *is oscillatory.*

Since (3) implies meas $E(M, T) = \infty$ whenever $M, T > 0$, the sufficiency of (3) for the oscillatory nature of (1) is a consequence of (*). In fact, the proof of (*) will depend upon a refinement of the argument used by Wintner in [2] in obtaining the criterion (3).

It is known that if (3) is relaxed to

Received by the editors February 23, 1955.

[1] This work was supported by the National Science Foundation research grant NSF-G481.

$$\limsup T^{-1}\int_0^T F(t)dt = \infty, \qquad T \to \infty, \tag{5}$$

then (1) need not be oscillatory; (see (II bis) of [1, p. 390]). It will be shown however, as a corollary of (*), that this situation cannot occur if, for instance, $f(t)$, or even $F(t)$, is bounded exponentially from below. Thus,

(**) *If, in addition to* (5), *the function* $F(t)$ *of* (2) *also satisfies*

$$F(t) > -\exp(Ct), \tag{6}$$

for some positive constant C, *then the equation* (1) *is oscillatory.*

2. **Proof of** (*). If $x=x(t)$ and $y=y(t)$ denote two linearly independent solutions of (1), it is clear from [2] that the equation (1) is oscillatory if and only if

$$\int_0^\infty (x^2+y^2)^{-1}dt = \infty. \tag{7}$$

In order to prove (*), suppose, if possible, that (1) is nonoscillatory. It will be shown that this assumption implies (7), hence a contradiction, and the proof of (*) will be complete. Since, for large values of t, the logarithmic derivative, z, of a solution of (1) satisfies the Riccati equation $z'+z^2+f=0$, the inequality $x^2+y^2 \leqq \text{const.} \exp(-2\int_0^t F(s)ds+Kt)$, in which K denotes a constant, holds for $0 \leqq t < \infty$; cf. formula line (7) of [2]. Consequently,

$$\int_0^\infty (x^2+y^2)^{-1}dt \geqq \text{const.} \int_0^\infty \exp[(2G-K)t]dt, \tag{8}$$

where $G=G(t)$ is defined by (3). Let $M>K$. In view of (8) and the inequalities

$$\int_0^\infty \exp[(2G-K)t]dt \geqq \int_T^\infty \exp[(2G-K)t]dt$$
$$\geqq \exp[(2M-K)T]\,\text{meas}\,E(M,T),$$

it follows that

$$\int_0^\infty (x^2+y^2)^{-1}dt \geqq \text{const.} \exp(MT)\,\text{meas}\,E(M,T). \tag{9}$$

Since the left side of the inequality (9) is independent of M and T, relation (4) implies (7). This completes the proof of (*).

3. **Proof of (**).** In view of (6) and the relation $(tG)'=F$, the inequality

$$(tG)' > -\exp(Ct) \tag{10}$$

holds for some positive constant C. If $a \leqq t \leqq b$ and $G(a)>0$, a quadrature of (10) leads to $tG(t)-aG(a) > -\int_a^t \exp(Cs)ds > -(b-a)\cdot\exp(Cb)$, and hence

$$G(t) > ab^{-1}G(a) - a^{-1}(b-a)\exp(Cb), \qquad a \leqq t \leqq b. \tag{11}$$

According to (5), there exists a sequence $t=t_1<t_2<\cdots$ such that $t_n\to\infty$ and $G(t_n)\to\infty$ as $n\to\infty$. For a given $M>0$, choose $a=t_n$ (for some n depending on M) such that $G(t_n)>2M$, and let b be defined by $b-a=\exp(-Cb)$. Then relation (11) implies $G(t)>2ab^{-1}M-a^{-1}$; hence, since $b-a\to 0$ as $a\to\infty$, $G(t)>M$ for $a\leqq t\leqq b$ and a sufficiently large. Consequently, the inequality

$$\exp(Ma)\,\text{meas}\,E(M, a) \geqq \exp(Ma - Cb)$$

holds for certain arbitrarily large numbers a and $b=a+\exp(-Cb)$. Clearly, for every fixed $M>C$, $\exp(MT)$ meas $E(M, T)\to\infty$ for a sequence of $T(=a)$ values tending to ∞. In particular, relation (4) holds and (*) now implies (**).

References

1. P. Hartman, *On nonoscillatory linear differential equations of second order*, Amer. J. Math. vol. 74 (1952) pp. 389–400.

2. A. Wintner, *A criterion of oscillatory stability*, Quarterly of Applied Mathematics vol. 7 (1949) pp. 115–117.

Purdue University