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PROJECTIONS IN THE SPACE (m)[1]

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A *projection* in a Banach space is a continuous linear mapping P of the space into itself which is such that $P^2=P$. Two closed linear manifolds M and N of a Banach space B are said to be *complementary* if each z of B is uniquely representable as $x+y$, where x is in M and y in N. This is equivalent to the existence of a projection for which M and N are the range and null space [7, p. 138]. It is therefore also true that closed linear subsets M and N of B are complementary if and only if the linear span of M and N is dense in B and there is a number $\epsilon>0$ such that $\|x+y\| \geqq \epsilon\|x\|$ if x is in M and y in N.

It is known that a Banach space M is complemented in each Banach space in which it can be embedded if it is isomorphic with a complemented subspace of the space (m) of bounded sequences. In particular, if M is a subspace of a Banach space Z and is isometric with a subspace M' of (m), then there is a projection of Z onto M of norm less than or equal to λ if there is a projection of (m) onto M' of norm equal to λ (see [8, p. 538] and [9, p. 945]). Thus the existence of a complement in (m) for a subspace M of (m) is independent of the method by which M is embedded in (m). Any separable Banach space is isometric with a subspace of (m) [3, p. 107]. Hence a separable Banach space is complemented in each space in which it can be em-

Presented to the Society, October 30, 1954; received by the editors February 4, 1955.

[1] Research supported in part by a grant from the National Science Foundation.

bedded if and only if it is complemented in (m).

It has been conjectured that no separable subspace of (m) is complemented in (m), or perhaps that separable complemented subspaces of (m) are reflexive (see [**9**, p. 946] and [**10**]). A partial answer to this is given in this paper. There are two classes of separable Banach spaces which are known to be not complemented in (m). These are the separable Banach spaces which have a subspace isomorphic with (c_0) [**9**, p. 946] and the separable Banach spaces whose first conjugate space is not weakly complete, of which l_1 is an example [**3**, p. 111].

It is interesting to note that (c_0) is complemented in any separable Banach space B in which it can be embedded. In particular, the projection of B onto (c_0) can be of norm 2 [**9**, p. 946], but there are separable Banach spaces containing (c_0) for which the norm of a projection onto (c_0) must have norm as large as 2 [**11**, p. 547].

It is also known [**1**] that if B is any one of the spaces L_p or l_p for $1<p<\infty$, then B can be embedded in a space Z such that no projection of Z onto B is of norm 1. In fact, if there is a projection of norm 1 of (m) onto a separable subspace M, then M has the Hahn-Banach extension property (and can easily be shown to have either of the properties discussed in [**4**, p. 92]). But it then follows from [**6**] that M is the space of continuous functions on an extremally disconnected compact Hausdorff space T (a compact Hausdorff space T for which the closures of open sets are open, or, equivalently, the closures of two disjoint open sets are disjoint). Such a Banach space M can not be separable unless T has only a finite number of points, since otherwise M has a subspace isometric with (m). To show this, divide T into two disjoint sets which are both open and closed. Continuing this process, form an infinite sequence of pair-wise disjoint sets U_i, each of which is both open and closed. Let $U=\bigcup_1^\infty U_i$ and V be the complement of $\overline{U}$. Let X be the normed linear space of functions defined on $U\cup V$ which are constant on U_i for each i, identically zero on V, and whose range consists of a finite number of real numbers. For each f of X, let $\|f\|=\sup|f(x)|$. For an arbitrary f of X, let $\alpha_1, \cdots, \alpha_k$ be the range of f. Let π_i be the subset of U on which $f(x)\equiv\alpha_i$, $i=1, 2, \cdots, k$. Then $\pi_1, \cdots, \pi_k$ are disjoint open sets. The sets $\bar{\pi}_1, \cdots, \bar{\pi}_k$ are also disjoint open sets, and $\bigcup_1^k\bar{\pi}_i=\overline{U}$. Define an extension $\bar{f}$ of f to all of B by letting $\bar{f}(x)=\alpha_i$ if $x\in\bar{\pi}_i$. Then $\bar{f}$ is continuous on B. The space X' of such extensions is isometric with X, for if $\|\bar{f}+\bar{f}'\|=|\bar{f}(x)+\bar{f}'(x)|$, where $x\in(\bar{\pi}_i\cap\bar{\pi}_j')$, then $\pi_i\cap\pi_j'\neq 0$ and $\|f+f'\|=|f(y)+f'(y)|=\|\bar{f}+\bar{f}'\|$ if $y\in(\pi_i\cap\pi_j')$. The completion of X' is clearly isometric with (m).

THEOREM 1. *Let M be a separable subspace of the space (m). If P is a projection of (m) onto M, then $\|P\|>1$.*

It is known [5, p. 519] that a Banach space with a basis $\{x_i\}$ is reflexive if and only if the following two conditions are satisfied:

(1) $\sum_1^\infty a_i x_i$ is convergent whenever $\|\sum_1^n a_i x_i\|$ is a bounded function of n;

(2) $\lim_{n\to\infty} \|f\|_n = 0$ for each linear functional f, where $\|f\|_n$ is the norm of f on the linear span of $x_{n+1}, x_{n+2}, \cdots$.

If the basis is unconditional, then (1) is satisfied if no subspace is isomorphic with (c_0) and (2) is satisfied if no subspace is isomorphic with l_1 [5, pp. 520–521]. These ideas are used in the following theorem.

THEOREM 2. *If M is a separable Banach space which is complemented in (m), then M is reflexive if M has an unconditional basis.*

PROOF. Let $\{x_n\}$ be a basis for M. If M is complemented in (m), then M can not have a subspace isomorphic with (c_0). Hence (1) is satisfied. If $\{x_n\}$ is an unconditional basis, then there is a positive number ϵ such that $\|\sum_1^n a_{p_i}x_{p_i} + \sum_1^m a_{q_i}x_{q_i}\| \geqq \epsilon \|\sum_1^n a_{p_i}x_{p_i}\|$ for any numbers a_i and integers m and n, provided the sets of integers $\{p_i\}$ and $\{q_i\}$ do not overlap (see [5, p. 518]). Now suppose that there is a linear functional f for which $\lim_{n\to\infty} \|f\|_n \neq 0$, where $\|f\|_n$ is the norm of f on the linear span of $x_{n+1}, x_{n+2}, \cdots$. Choose a number $\sigma>0$ and a sequence z_k with $z_k = \sum_{i=n_{k-1}}^{n_k-1} a_i x_i$, $1=n_0<n_1<n_2<\cdots$, $\|z_k\|=1$, and $f(z_k)>\sigma$ for each k. Let N_k be the null space of f on the linear span X_k of $x_{n_{k-1}}, \cdots, x_{n_k-1}$. Since the linear span of z_k and N_k is X_k, it follows that finite sums of type $(\sum a_i z_i)+(\sum u_i)$, where $u_i \in N_i$, are dense in M. Let R be the closed linear span of the elements $\{z_k\}$ and S be the closed linear span of the union of the sets $\{N_k\}$. Then, if $\sum_+$ and $\sum_-$ denote sums over indices i for which $a_i \geqq 0$ and $a_i<0$, respectively, we have:

$$\begin{aligned}\|\sum a_i z_i + \sum u_i\| &\geqq \epsilon \|\sum_+ a_i z_i + \sum_+ u_i\| \\ &\geqq \epsilon \left| f(\sum_+ a_i z_i + \sum_+ u_i)\right| / \|f\| \\ &= \epsilon \left| f(\sum_+ a_i z_i)\right| / \|f\| \geqq \epsilon\sigma(\sum_+ |a_i|)/\|f\|.\end{aligned}$$

Likewise, $\|\sum a_i z_i + \sum u_i\| \geqq \epsilon\sigma(\sum_- |a_i|)/\|f\|$. Therefore

$$\|\sum a_i z_i + \sum u_i\| \geqq 2^{-1}\epsilon\sigma(\sum |a_i|)/\|f\| \geqq 2^{-1}\epsilon\sigma\|\sum a_i z_i\|/\|f\|.$$

Therefore R and S are complementary in M. However, $\|\sum a_i z_i\| \leqq \sum |a_i|$ and, from the above, $\|\sum a_i z_i\| \geqq 2^{-1}\epsilon\sigma(\sum |a_i|)/\|f\|$. Therefore R is isomorphic with l_1. But if M is complemented in (m), then l_1

is complemented in (m). Since l_1 is not complemented in (m), this is contradictory and therefore (2) above is satisfied for the basis $\{x_i\}$ of M. Therefore M is reflexive.

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