THE EMBEDDING OF HOMEOMORPHISMS IN FLOWS

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1. Introduction. Let $X$ be a topological space, and let $R$ be the real number system. If $F$ is a function on $X \times R$, $x \in X$ and $t \in R$, we will usually denote $F(x, t)$ by $F_t(x)$. For each real number $t$, $F_t$ denotes the obvious function on $X$.

A (topological) flow on $X$ is a continuous function $F$ on $X \times R$ into $X$ such that:

1. $F_t$ is a homeomorphism of $X$ onto $X$ for each $t \in R$; and
2. $F_{t+s}(x) = F_t(F_s(x))$ for all $t, s \in R$ and $x \in X$.

We now state the general embedding problem for flows.

**Embedding Problem.** For a given space $X$ and a given homeomorphism $f$ of $X$ onto $X$, does there exist a flow $F$ on $X$ for which $F_1 = f$?

If such a flow $F$ exists, we say that $f$ is embedded in $F$.

In general, the embedding problem is quite difficult. In this paper we discuss only the case in which $X$ is an interval of real numbers.

If $X$ is an interval of real numbers and $f$ is a continuously differentiable homeomorphism of $X$ onto $X$, we may ask whether or not $f$ can be embedded in a flow $F$ for which each homeomorphism $F_t$ has a continuous derivative. We obtain some results pertaining to the solution of this latter problem, although a complete solution is not obtained.

2. The embedding problem for intervals. Let $f$ be a homeomorphism of an interval of real numbers onto itself. In order that it be possible to embed $f$ in a flow, it is obviously necessary that $f$ be order preserving. We prove that this condition is also sufficient.

**Lemma 1.** If $h$ is a homeomorphism of a closed interval $[a, b]$ onto itself such that $a$ and $b$ are the only fixed points of $h$, then it is possible to embed $h$ in a flow $H$ such that if $a < x < b$ and $-1 < t < 1$ then $H_t(x)$ is between $h^{-1}(x)$ and $h(x)$.

**Proof.** It is proved in [1] and [2] that there exists an order preserving homeomorphism $\psi$ on $[a, b]$ onto $[0, \infty]$ and a positive number $A$ such that $h(x) = \psi^{-1}(A\psi(x))$ for $a \leq x \leq b$. If $f(x) > x$ for $a < x < b$, present to the Society, November 27, 1954; received by the editors February 14, 1955.

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then \( A > 1 \); if \( f(x) < x \) for \( a < x < b \), then \( A < 1 \). If we now define \( H_t(x) = \psi^{-1}(A^t \psi(x)) \) for \( a \leq x \leq b \) and \( t \in \mathbb{R} \), then it is easy to verify that \( H \) has the desired properties.

**THEOREM 1.** If \( f \) is an order preserving homeomorphism of an interval \( J \) onto itself, then it is possible to embed \( f \) in a flow \( F \).

**Proof.** We may assume without loss of generality that \( J \) is a closed interval. Let \( K \) be the set of all fixed points of \( f \), and let \( \Sigma \) be the set of all closed intervals which are the closures of the components of \( J - K \). For each interval \( u \in \Sigma \), we use Lemma 1 to obtain a flow \( U \) on \( J \) such that \( U_t = f \upharpoonright u \), and such that for \(-1 < t < 1 \) and each point \( x \) that is interior to \( U \), \( U_t(x) \) is between \( f^{-1}(x) \) and \( f(x) \).

Now let \( x \) be a member of \( J \). If \( x \in K \), we define \( F_t(x) = x \) for all \( t \in \mathbb{R} \). If \( x \in J - K \), then there exists \( u \in \Sigma \) such that \( x \in u \); and in this case we define \( F_t(x) = U_t(x) \) for all \( t \in \mathbb{R} \).

For each \( t \), \( F_t \) is a one-to-one order preserving function on \( J \) onto itself, and it follows that \( F_t \) is a homeomorphism. Moreover, if \( t \in \mathbb{R} \), \( s \in \mathbb{R} \) and \( x \in J \), then it is easy to verify that \( F_t(F_s(x)) = F_{t+s}(x) \).

We must finally prove that \( F \) is continuous on \( J \times \mathbb{R} \). Since each \( F_t \) is continuous and \( F_tF_s = F_{t+s} \) for all \( t, s \), it is sufficient to prove that \( F \) is continuous at each point of the form \((a, 0)\), \( a \in J \). If \( a \in J - K \), it is obvious that \( F \) is continuous at \((a, 0)\). Thus, let us assume that \( a \in K \) and that \((x_n, t_n) \to (a, 0)\) as \( n \to \infty \). For all large values of \( n \) we have \(-1 < t_n < 1 \), and hence \( F(x_n, t_n) \) is between \( f^{-1}(x_n) \) and \( f(x_n) \). Since \( f \) and \( f^{-1} \) are continuous and \( f(a) = f^{-1}(a) = F(a, 0) \), it follows that \( F(x_n, t_n) \to F(a, 0) \) as \( n \to \infty \). Thus \( F \) is continuous.

If the function \( h \) of the lemma has a continuous derivative, then the function \( \psi \) may be chosen so as to have a continuous derivative on the open interval \((a, b)\). It follows that each function \( H_t \) has a continuous derivative on the open interval \((a, b)\). Turning now to the above theorem, we see that if \( f \) has a continuous derivative on \( J \), then the flow \( F \) may be constructed so that each \( F_t \) has a continuous derivative on \( J - K \). We have no assurance, however, that the functions \( F_t \) will have derivatives at fixed points of \( f \), even though \( f \) has a derivative at these points. An interesting problem is that of determining conditions for \( f \) that will guarantee that we can construct the flow \( F \) in such a manner that each function \( F_t \) has a continuous derivative on all of \( J \). We obtain a few results pertaining to this problem in the next section.

3. **Flows of continuously differentiable functions.** Throughout this section we assume that \( f \) is a function which has the following properties:
(i) \( f \) is a homeomorphism of a half open interval \((a, b]\) onto itself;
(ii) \( f \) has a continuous derivative on \((a, b]\);
(iii) \( f(x) > x \) for \( a < x < b \);
(iv) \( f'(x) > 0 \) for \( a < x \leq b \);
(v) \( f' \) is monotone nonincreasing on the interval \((a, b]\).

We are going to prove that there exists a unique flow \( F \) such that
\( F_t = f \) and such that \( F_t \) has a continuous derivative on \((a, b]\) for each
real number \( t \). It is clear that if such a flow exists, then \( f \) must com-
mute with \( F_t \) for each \( t \). Thus, in trying to construct the flow \( F \), it is
reasonable to first try to determine the set of all continuously differ-
entiable order preserving homeomorphisms that commute with \( f \).

Using conditions (i)–(v) above, it is possible to prove the following
lemma. Since the proof is fairly straightforward, it is omitted.

**Lemma 2.** If \( a < x < b \) and \( a < y < b \), then the infinite product
\[
\prod_{n=0}^{\infty} \left[ f'(f^n(x))/f'(f^n(y)) \right]
\]
converges. Moreover, if \( a < a* < b* < b \), then for each fixed \( y \), the infinite
product converges uniformly in \( x \) for \( a* \leq x \leq b* \).

In view of the above lemma, we obtain a continuous function \( \phi \) if
we define \( c = (a+b)/2 \) and then define \( \phi(x) = \prod_{n=0}^{\infty} [f'(f^n(x))/f'(f^n(c))] \)
for \( a < x < b \).

**Theorem 2.** If \( g \) is a continuously differentiable homeomorphism of
\((a, b] \) onto itself and \( g \) commutes with \( f \), then
\( g'(x) = g'(b)\phi(x)/\phi(g(x)) \)
for \( a < x < b \).

**Proof.** Since \( f \) and \( g \) commute, we obtain for all \( x \),
\( f(g(x)) = g(f(x)) \).

We now differentiate each side of the above equation, obtaining
\( f'(g(x))g'(x) = g'(f(x))f'(x) \).

We solve for \( g'(x) \), obtaining
\( g'(x) = [f'(x)/f'(g(x))]g'(f(x)) \).

Next we replace \( x \) by \( f^k(x) \) in the above equation, and use the fact that \( g \) and \( f^k \) commute. We obtain
\( g'(f^k(x)) = [f'(f^k(x))/f'(f^k(g(x)))]g'(f^{k+1}(x)) \).
If \( n \) is a positive integer, it follows that

\[
g'(x) = \left\{ \prod_{k=0}^{n} \frac{f'(f^k(x))}{f'(f^k(g(x)))} \right\} g'(f^{n+1}(x)).
\]

Since \( g' \) is continuous at \( b \) and \( \lim_{n \to \infty} f^{n+1}(x) = b \), we obtain

\[
g'(x) = \left\{ \prod_{k=0}^{\infty} \frac{f'(f^k(x))}{f'(f^k(g(x)))} \right\} g'(b) = g'(b) \frac{\phi(x)}{\phi(g(x))}.
\]

**Corollary 1.** Under the same hypotheses as for the above theorem, \( g'(b) \neq 0 \).

It is easy to prove under the hypotheses for \( f \) that \( 0 < f'(b) < 1 \). We define \( A = f'(b) \). It follows from the above results that any continuously differentiable homeomorphism on \((a, b]\) that commutes with \( f \) must be a solution of a differential equation of the form

\[
dy/dx = A^{-1} \phi(x) / \phi(y).
\]

We now study the existence and uniqueness of solutions of the differential equation

\[
E(t): \quad dy/dx = A^t \phi(x) / \phi(y).
\]

**Lemma 3.** Suppose that \( t \) is a real number and that \( a \leq d < b \). Then there exists at most one continuous function \( g \) on \((d, b]\) such that \( g \) satisfies \( E(t) \) on \((d, b)\) and \( g(b) = b \).

**Proof.** Suppose that \( g \) and \( h \) are continuous on \((d, b]\), both are solutions of \( E(t) \) on the interval \((d, b)\), and \( g(b) = h(b) = b \). If \( g \) and \( h \) are not identical, then there exists a point \( p \) in the interval such that \( g(p) \neq h(p) \). We may assume without loss of generality that \( g(p) > h(p) \). There exists a point \( q \), \( p < q < b \), such that \( g(q) = h(q) \) and \( g(x) > h(x) \) for \( p < x < q \). We define \( w(x) = g(x) - h(x) \). The Mean Value Theorem yields a point \( r \), \( p < r < q \), for which

\[
\frac{g(p) - h(p)}{p - q} = \frac{w(p) - w(q)}{p - q} = w'(r) = g'(r) - h'(r)
\]

\[
= A^t \phi(r) \left\{ \frac{1}{\phi(g(r))} - \frac{1}{\phi(h(r))} \right\}.
\]

Since \( g(r) > h(r) \) and \( \phi \) is nonincreasing, it follows that \( A^t \phi(r) \left\{ 1/\phi(g(r)) - 1/\phi(h(r)) \right\} \) is nonnegative. We now have a contradiction, since \( [g(p) - h(p)]/(p - q) \) is negative.

We next define a function \( \Phi \) on the interval \((a, b]\) by letting

\[
\Phi(x) = \int_a^x \phi(t) \, dt.
\]
Lemma 4. \( \Phi \) is an order reversing homeomorphism of \((a, b]\) onto \([0, \infty)\).

Proof. It is obvious that \( \Phi \) is continuous and strictly monotone decreasing, and that \( \Phi(b) = 0 \). Hence it is sufficient to prove that \( \lim_{x \to a} \Phi(x) = \infty \).

Let us assume that \( a < x < c \). We first observe that
\[
\Phi(x) \geq \int_x^c \phi(t)dt.
\]
However, for \( x \leq t \leq c \), \( \phi(t) \geq \prod_{k=0}^n \left[ f'(f^k(t))/f'(f^k(c)) \right] \) for each \( n \). Moreover, it is easy to prove by induction on \( n \) that
\[
\prod_{k=0}^n f'(f^k(t)) = \frac{d}{dt} f^{n+1}(t).
\]
It follows from these facts that \( \phi(x) \geq \left[ f^{n+1}(c) - f^{n+1}(x) \right] / \prod_{k=0}^n f'(f^k(c)) \) for every \( n \). Now let \( B \) be any positive number. It is easy to see that
\[
\lim_{n \to \infty} f^{n+1}(c) = b \quad \text{and} \quad \lim_{n \to \infty} \prod_{k=0}^n f'(f^k(c)) = 0.
\]
Thus there exists an integer \( m \) such that \( f^{m+1}(c) > (a+2b)/3 \) and
\[
\prod_{k=0}^m f'(f^k(c)) < (b - a)/3B.
\]
Now choose \( \varepsilon > 0 \) such that if \( a < x < a+\varepsilon \) then \( f^{m+1}(x) < (2a+b)/3 \). It now follows that if \( a < x < a+\varepsilon \) then \( \Phi(x) > B \). Therefore,
\[
\lim_{x \to a} \Phi(x) = \infty.
\]

Lemma 5. If \( g \) is an order preserving homeomorphism of \((a, b]\) onto itself and \( g \) commutes with \( f \), then \( \lim_{x \to b} \phi(x)/\phi(g(x)) = 1 \).

Proof. It is easy to verify that \( \lim_{x \to b} \phi(x)/\phi(f^m(x)) = 1 \) if \( m \) is an integer. We shall show that there exists an integer \( n \) such that \( f^{n+1}(c) \leq g(x) \leq f^{n+2}(x) \) for \( a < x < b \). Since \( \phi \) is monotone, our lemma will follow from the inequalities
\[
\phi(x)/\phi(f^{n+1}(x)) \leq \phi(x)/\phi(g(x)) \leq \phi(x)/\phi(f^{n+2}(x)).
\]
There exists an integer \( n \) such that \( f^n(c) \leq g(c) \leq f^{n+1}(c) \). Now let \( x \) be any member of the interval \((a, b)\). There exists an integer \( k \) such that \( f^k(c) \leq x \leq f^{k+1}(c) \). If we now use the fact that each power of \( f \) is monotone increasing and commutes with \( g \), we obtain
\[
f^{n-k}(x) \leq f^{n+k}(c) \leq g(f^k(c)) \leq g(x) \leq g(f^{k+1}(c)) \leq f^{n+k+2}(c) \leq f^{n+2}(x).
\]
Thus, \( f^{n-1}(x) \leq g(x) \leq f^{n+2}(x) \), and our lemma follows.

**Theorem 3.** There exists a unique flow \( F \) on \((a, b]\) such that \( F_1 = f \) and such that each \( F_t \) is continuously differentiable on \((a, b]\). Moreover, if \( g \) is any continuously differentiable homeomorphism of \((a, b]\) onto itself such that \( g \) commutes with \( f \), then there exists a real number \( t \) such that \( F_t = g \).

**Proof.** We define \( F_t(x) = \Phi^{-1}(A^t \Phi(x)) \) for \( a < x \leq b \). It is easy to verify that each function \( F_t \) is a homeomorphism of \((a, b]\) onto itself. Moreover, it is easy to prove that \( F_s F_t = F_{s+t} \) for all \( s \) and \( t \), and that \( F \) is continuous. Thus \( F \) is a flow on the interval \((a, b]\).

Since \( \Phi'(x) = -\phi(x) \), it follows that \( F_t'(x) = A^t \phi(x)/(\phi(F_t(x)) \) and hence \( F_t \) satisfies the differential equation \( E(t) \) on the interval \((a, b]\). Since \( f \) obviously satisfies \( E(1) \), it follows from Lemma 3 that \( F_1 = f \).

Since \( F_1 = f \) and \( F \) is a flow, it follows that \( F_t \) commutes with \( f \) for each \( t \). We now use Lemma 5 to obtain \( \lim_{x \to b} F_t'(x) = A^t \). It follows that \( F_t \) is differentiable at \( b \), and that \( F_t'(b) = A^t \). Thus each function \( F_t \) has a continuous derivative on \((a, b]\).

Let us now prove that the flow \( F \) is unique. We observe that if \( n \) is a positive integer and \( g \) is a continuously differentiable homeomorphism of \((a, b]\) onto itself for which \( g^n = f \), then

\[
f'(x) = g'(g^{n-1}(x))g'(g^{n-2}(x)) \cdots g'(x).
\]

If we let \( x \) tend to \( b \), it is easy to see that we obtain

\[
f'(b) = (g'(b))^n.
\]

It follows that \( g \) satisfies the equation \( E(1/n) \). Now let \( G \) be any flow of continuously differentiable functions which satisfies \( G_1 = f \). We see that \( G_{1/n} \) and \( F_{1/n} \) both satisfy \( E(1/n) \) for each positive integer \( n \) and hence \( G_{1/n} = F_{1/n} \). It follows that \( G_r = F_r \) for every rational number \( r \). Since flows are continuous in both variables, \( G_t(x) = F_t(x) \) for all real \( t \) and all \( x \) in the interval \((a, b]\). Therefore \( F \) is unique.

Now suppose that \( g \) is a continuously differentiable homeomorphism of \((a, b]\) onto itself and that \( g \) commutes with \( f \). It follows from Corollary 1 that \( g'(b) \neq 0 \). Thus, since \( g \) is order preserving, \( g'(b) > 0 \) and there exists a real number \( t \) such that \( A^t = g'(b) \). It follows from Theorem 2 that \( g \) satisfies the differential equation \( E(t) \), and since \( F_t \) also satisfies \( E(t) \), Lemma 3 implies that \( F_t = g \).

It is obvious that we may extend the domain of \( f \) and of the functions \( F_t \) to the closed interval \([a, b]\) by defining \( F_t(a) = a \), and we shall then obtain a flow \( F \) on \([a, b]\). However, as is shown by the following example, it is not necessarily true that each of the functions
$F_t$ will have a derivative at $a$, even though $f$ has a derivative at $a$.

**Example.** Let $g(x) = 4x/(3x+1)$ for $0 \leq x \leq 1$. Next define a function $f$ such that $f(x) = 2x$ for $0 \leq x \leq 1/3$; $f(x) = g(x)$ for $1/2 \leq x \leq 1$; $f(x)$ arbitrary for $1/3 \leq x \leq 1/2$, subject only to the requirement that $f$ satisfy the conditions (i)-(v) listed at the beginning of §3. It is easily seen that it is possible to define such a function $f$. Theorem 3 implies that there exist flows $G$ and $F$ such that $G_1 = g$, $F_1 = f$ and, all of the functions $F_t$ and $G_t$ are continuously differentiable on $(0, 1]$. If one makes use of the uniqueness of $G$, it is easily verified that $G_{1/2}(x) = 4^{1/2}x/((4^{1/2} - 1)x + 1)$. Since $g(x) = f(x)$ for $1/2 \leq x \leq 1$, it is easily seen that $G_{1/2}$ and $F_{1/2}$ satisfy the same differential equation on the interval $1/2 \leq x \leq 1$. It follows from Lemma 3 that $F_{1/2}(x) = G_{1/2}(x)$ for $1/2 \leq x \leq 1$, and we use this fact to compute $F_{1/2}(1/2) = 2/3$. Next we define points $p_0$, $p_1$, $p_2$, · · · by letting $p_0 = 1/2$, $p_1 = 1/3$, and $p_n = F_{-1}(p_{n-2})$ for $n \geq 2$. Since $F_{1/2}(p_0) = 2/3 = F_1(p_1)$, we obtain $F_{1/2}(p_1) = p_0$. It is easy to prove by induction that $F_{1/2}(p_n) = p_{n-1}$ for every positive integer $n$. Since $F_{-1}(x) = x/2$ for $0 \leq x \leq 1/2$, it is easy to compute the values of the numbers $p_n$, and to verify that the difference quotient $(F_{1/2}(p_n) - F_{1/2}(0))/(p_n - 0)$ is equal to $3/2$ if $n$ is odd and is equal to $4/3$ if $n$ is even. Thus $F_{1/2}$ does not have a derivative at 0.

This example demonstrates that it is possible to have a function $f$ which is continuously differentiable on the closed interval $[a, b]$ and which satisfies conditions (i)-(v) on the half-open interval $(a, b]$, but which cannot be embedded in a flow of functions that are continuously differentiable on the closed interval $[a, b]$.

We do, however, have the following result.

**Theorem 4.** If $f$ is continuously differentiable over the closed interval $[a, b]$ and satisfies (i)-(v) on $(a, b)$, then the set $G$ of all continuously differentiable order preserving homeomorphisms $g$ of $[a, b]$ onto itself for which $g$ commutes with $f$ forms a group which is isomorphic to either the group of real numbers or the group of integers.

**Proof.** We use Theorem 3 to obtain a flow $F$ on $[a, b]$ such that each of the functions $F_t$ has a continuous derivative on $(a, b]$, and $F_1 = f$. It follows from Theorem 3 that if $g \in G$, then there exists a real number $t$ such that $F_t = g$. We let $S$ be the set of all real numbers $t$ for which $F_t \in G$. If we use the fact that $F_{t+s}(x) = F_t(F_s(x))F'_s(x)$ for $a \leq x \leq b$, it is easy to see that if $t$ and $s$ are members of $S$ then $t + s$ is also a member of $S$. Moreover, if $t$ is in $S$ then $-t$ is in $S$. Thus $S$ is a subgroup of the reals. It is not difficult to prove that $G$ is a group which is isomorphic to $S$, and consequently our theorem is
proved if we can show that either \( S \) is the set \( R \) of all real numbers or else \( S \) is a nondegenerate discrete subgroup of \( R \).

We observe that \( S \) is nondegenerate since \( 1 \in S \). Now assume that \( S \) is isomorphic to neither \( R \) nor the group of integers. It then follows that both \( S \) and \( R - S \) are dense in \( R \), and this implies that \( R - S \) is uncountable. We shall now show that this is impossible. In order to accomplish this, we define \( u_t \) to be the lower derivate of \( F_t \) at \( a \), and we define \( v_t \) to be the upper derivate of \( F_t \) at \( a \). It is easily seen that if \( s < t \), then \( F_s(x) < F_t(x) \) for \( a < x < b \), and it follows that if \( s < t \) then \( u_s \leq u_t \) and \( v_s \leq v_t \). Now let \( r \) and \( t \) be members of \( R - S \), \( r < t \). Since \( S \) is dense in \( R \), there exists \( s \in S \) such that \( r < s < t \). We see that \( u_s \leq v_r \leq u_s \leq u_t < v_t \). Thus, by defining \( L(t) \) to be the open interval \((u_t, v_t)\) for each \( t \in R - S \), we obtain a one-to-one function \( L \) that maps \( R - S \) onto a set of mutually exclusive open intervals. This implies that \( R - S \) is countable, and we have a contradiction.

We conclude by listing two unsolved problems that are of interest.

**Problem 1.** Find conditions on \( f \) that are necessary and sufficient for \( S = R \), where \( S \) and \( R \) are the sets defined in the proof of Theorem 4.

**Problem 2.** Replace condition (v) on \( f \) by a weaker condition.

The solution of these problems would constitute a major step toward obtaining necessary and sufficient conditions that it be possible to embed a continuously differentiable homeomorphism (having arbitrarily many fixed points) of an interval onto itself in a flow of continuously differentiable homeomorphisms.

**Bibliography**


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