THE EMBEDDING OF HOMEOMORPHISMS IN FLOWS

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1. **Introduction.** Let X be a topological space, and let R be the real number system. If F is a function on $X \times R$, $x \in X$ and $t \in R$, we will usually denote $F(x, t)$ by $F_t(x)$. For each real number t, F_t denotes the obvious function on X.

A (*topological*) *flow* on X is a continuous function F on $X \times R$ into X such that:

(1) F_t is a homeomorphism of X onto X for each $t \in R$; and

(2) $F_t(F_s(x)) = F_{t+s}(x)$ for all $t \in R$, $s \in R$ and $x \in X$.

We now state the general embedding problem for flows.

EMBEDDING PROBLEM. *For a given space X and a given homeomorphism f of X onto X, does there exist a flow F on X for which $F_1 = f$?*

If such a flow F exists, we say that f is *embedded* in F.

In general, the embedding problem is quite difficult. In this paper we discuss only the case in which X is an interval of real numbers.

If X is an interval of real numbers and f is a continuously differentiable homeomorphism of X onto X, we may ask whether or not f can be embedded in a flow F for which each homeomorphism F_t has a continuous derivative. We obtain some results pertaining to the solution of this latter problem, although a complete solution is not obtained.

2. **The embedding problem for intervals.** Let f be a homeomorphism of an interval of real numbers onto itself. In order that it be possible to embed f in a flow, it is obviously necessary that f be order preserving. We prove that this condition is also sufficient.

LEMMA 1. *If h is a homeomorphism of a closed interval $[a, b]$ onto itself such that a and b are the only fixed points of h, then it is possible to embed h in a flow H such that if $a < x < b$ and $-1 < t < 1$ then $H_t(x)$ is between $h^{-1}(x)$ and $h(x)$.*

PROOF. It is proved in [1] and [2] that there exists an order preserving homeomorphism ψ on $[a, b]$ onto $[0, \infty]$ and a positive number A such that $h(x) = \psi^{-1}(A\psi(x))$ for $a \leq x \leq b$. If $f(x) > x$ for $a < x < b$,

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then $A>1$; if $f(x)<x$ for $a<x<b$, then $A<1$. If we now define $H_t(x)=\psi^{-1}(A^t\psi(x))$ for $a\leqq x\leqq b$ and $t\in R$, then it is easy to verify that H has the desired properties.

THEOREM 1. *If f is an order preserving homeomorphism of an interval J onto itself, then it is possible to embed f in a flow F.*

PROOF. We may assume without loss of generality that J is a closed interval. Let K be the set of all fixed points of f, and let Σ be the set of all closed intervals which are the closures of the components of $J-K$. For each interval $u\in\Sigma$, we use Lemma 1 to obtain a flow U on u such that $U_1=f|u$, and such that for $-1<t<1$ and each point x that is interior to U, $u_t(x)$ is between $f^{-1}(x)$ and $f(x)$.

Now let x be a member of J. If $x\in K$, we define $F_t(x)=x$ for all $t\in R$. If $x\in J-K$, then there exists $u\in\Sigma$ such that $x\in u$; and in this case we define $F_t(x)=U_t(x)$ for all $t\in R$.

For each t, F_t is a one-to-one order preserving function on J onto J, and it follows that F_t is a homeomorphism. Moreover, if $t\in R$, $s\in R$ and $x\in J$, then it is easy to verify that $F_t(F_s(x))=F_{t+s}(x)$.

We must finally prove that F is continuous on $J\times R$. Since each F_t is continuous and $F_tF_s=F_{t+s}$ for all t, s, it is sufficient to prove that F is continuous at each point of the form $(a, 0)$, $a\in J$. If $a\in J-K$, it is obvious that F is continuous at $(a, 0)$. Thus, let us assume that $a\in K$ and that $(x_n, t_n)\to(a, 0)$ as $n\to\infty$. For all large values of n we have $-1<t_n<1$, and hence $F(x_n, t_n)$ is between $f^{-1}(x_n)$ and $f(x_n)$. Since f and f^{-1} are continuous and $f(a)=f^{-1}(a)=F(a, 0)$, it follows that $F(x_n, t_n)\to F(a, 0)$ as $n\to\infty$. Thus F is continuous.

If the function h of the lemma has a continuous derivative, then the function ψ may be chosen so as to have a continuous derivative on the open interval (a, b). It follows that each function H_t has a continuous derivative on the open interval (a, b). Turning now to the above theorem, we see that if f has a continuous derivative on J, then the flow F may be constructed so that each F_t has a continuous derivative on $J-K$. We have no assurance, however, that the functions F_t will have derivatives at fixed points of f, even though f has a derivative at these points. An interesting problem is that of determining conditions for f that will guarantee that we can construct the flow F in such a manner that each function F_t has a continuous derivative on all of J. We obtain a few results pertaining to this problem in the next section.

3. **Flows of continuously differentiable functions.** Throughout this section we assume that f is a function which has the following properties:

(i) f is a homeomorphism of a half open interval $(a, b]$ onto itself;
(ii) f has a continuous derivative on $(a, b]$;
(iii) $f(x) > x$ for $a < x < b$;
(iv) $f'(x) > 0$ for $a < x \leqq b$;
(v) f' is monotone nonincreasing on the interval $(a, b]$.

We are going to prove that there exists a unique flow F such that $F_1 = f$ and such that F_t has a continuous derivative on $(a, b]$ for each real number t. It is clear that if such a flow exists, then f must commute with F_t for each t. Thus, in trying to construct the flow F, it is reasonable to first try to determine the set of all continuously differentiable order preserving homeomorphisms that commute with f.

Using conditions (i)–(v) above, it is possible to prove the following lemma. Since the proof is fairly straightforward, it is omitted.

LEMMA 2. *If $a < x < b$ and $a < y < b$, then the infinite product*

$$\prod_{n=0}^{\infty} [f'(f^n(x))/f'(f^n(y))]$$

converges. Moreover, if $a < a^* < b^* < b$, then for each fixed y, the infinite product converges uniformly in x for $a^* \leqq x \leqq b^*$.

In view of the above lemma, we obtain a continuous function ϕ if we define $c = (a+b)/2$ and then define $\phi(x) = \prod_{n=0}^{\infty} [f'(f^n(x))/f'(f^n(c))]$ for $a < x < b$.

THEOREM 2. *If g is a continuously differentiable homeomorphism of $(a, b]$ onto itself and g commutes with f, then*

$$g'(x) = g'(b)\phi(x)/\phi(g(x))$$

for $a < x < b$.

PROOF. Since f and g commute, we obtain for all x,

$$f(g(x)) = g(f(x)).$$

We now differentiate each side of the above equation, obtaining

$$f'(g(x))g'(x) = g'(f(x))f'(x).$$

We solve for $g'(x)$, obtaining

$$g'(x) = [f'(x)/f'(g(x))]g'(f(x)).$$

Next we replace x by $f^k(x)$ in the above equation, and use the fact that g and f^k commute. We obtain

$$g'(f^k(x)) = [f'(f^k(x))/f'(f^k(g(x)))]g'(f^{k+1}(x)).$$

If n is a positive integer, it follows that

$$g'(x) = \left\{ \prod_{k=0}^{n} \left[\frac{f'(f^k(x))}{f'(f^k(g(x)))} \right] \right\} g'(f^{n+1}(x)).$$

Since g' is continuous at b and $\lim_{n\to\infty} f^{n+1}(x) = b$, we obtain

$$g'(x) = \left\{ \prod_{k=0}^{\infty} \left[\frac{f'(f^k(x))}{f'(f^k(g(x)))} \right] \right\} g'(b) = g'(b) \frac{\phi(x)}{\phi(g(x))}.$$

COROLLARY 1. *Under the same hypotheses as for the above theorem,* $g'(b) \neq 0$.

It is easy to prove under the hypotheses for f that $0 < f'(b) < 1$. We define $A = f'(b)$. It follows from the above results that any continuously differentiable homeomorphism on $(a, b]$ that commutes with f must be a solution of a differential equation of the form

$$dy/dx = A^t \phi(x)/\phi(y).$$

We now study the existence and uniqueness of solutions of the differential equation

$$E(t): \quad dy/dx = A^t \phi(x)/\phi(y).$$

LEMMA 3. *Suppose that t is a real number and that $a \leqq d < b$. Then there exists at most one continuous function g on $(d, b]$ such that g satisfies $E(t)$ on (d, b) and $g(b) = b$.*

PROOF. Suppose that g and h are continuous on $(d, b]$, both are solutions of $E(t)$ on the interval (d, b), and $g(b) = h(b) = b$. If g and h are not identical, then there exists a point p in the interval such that $g(p) \neq h(p)$. We may assume without loss of generality that $g(p) > h(p)$. There exists a point q, $p < q \leqq b$, such that $g(q) = h(q)$ and $g(x) > h(x)$ for $p \leqq x < q$. We define $w(x) = g(x) - h(x)$. The Mean Value Theorem yields a point r, $p < r < q$, for which

$$\frac{g(p) - h(p)}{p - q} = \frac{w(p) - w(q)}{p - q} = w'(r) = g'(r) - h'(r)$$

$$= A^t \phi(r) \left\{ \frac{1}{\phi(g(r))} - \frac{1}{\phi(h(r))} \right\}.$$

Since $g(r) > h(r)$ and ϕ is nonincreasing, it follows that $A^t \phi(r) \{ 1/\phi(g(r)) - 1/\phi(h(r)) \}$ is nonnegative. We now have a contradiction, since $[g(p) - h(p)]/(p - q)$ is negative.

We next define a function Φ on the interval $(a, b]$ by letting $\Phi(x) = \int_x^b \phi(t)\,dt$.

LEMMA 4. *Φ is an order reversing homeomorphism of $(a, b]$ onto $[0, \infty)$.*

PROOF. It is obvious that Φ is continuous and strictly monotone decreasing, and that $\Phi(b)=0$. Hence it is sufficient to prove that $\lim_{x\to a}\Phi(x)=\infty$.

Let us assume that $a<x<c$. We first observe that

$$\Phi(x) \geqq \int_x^c \phi(t)dt.$$

However, for $x\leqq t\leqq c$, $\phi(t)\geqq \prod_{k=0}^n [f'(f^k(t))/f'(f^k(c))]$ for each n. Moreover, it is easy to prove by induction on n that

$$\prod_{k=0}^{n} f'(f^k(t)) = \frac{d}{dt} f^{n+1}(t).$$

It follows from these facts that $\phi(x)\geqq [f^{n+1}(c)-f^{n+1}(x)]/\prod_{k=0}^n f'(f^k(c))$ for every n. Now let B be any positive number. It is easy to see that $\lim_{n\to\infty} f^{n+1}(c)=b$ and $\lim_{n\to\infty}\prod_{k=0}^n f'(f^k(c))=0$. Thus there exists an integer m such that $f^{m+1}(c)>(a+2b)/3$ and

$$\prod_{k=0}^{m} f'(f^k(c)) < (b-a)/3B.$$

Now choose $\epsilon>0$ such that if $a<x<a+\epsilon$ then $f^{m+1}(x)<(2a+b)/3$. It now follows that if $a<x<a+\epsilon$ then $\Phi(x)>B$. Therefore,

$$\lim_{x\to a}\Phi(x) = \infty.$$

LEMMA 5. *If g is an order preserving homeomorphism of $(a, b]$ onto itself and g commutes with f, then $\lim_{x\to b}\phi(x)/\phi(g(x))=1$.*

PROOF. It is easy to verify that $\lim_{x\to b}\phi(x)/\phi(f^m(x))=1$ if m is an integer. We shall show that there exists an integer n such that $f^{n-1}(x)\leqq g(x)\leqq f^{n+2}(x)$ for $a<x<b$. Since ϕ is monotone, our lemma will follow from the inequalities

$$\phi(x)/\phi(f^{n-1}(x)) \leqq \phi(x)/\phi(g(x)) \leqq \phi(x)/\phi(f^{n+2}(x)).$$

There exists an integer n such that $f^n(c)\leqq g(c)\leqq f^{n+1}(c)$. Now let x be any member of the interval (a, b). There exists an integer k such that $f^k(c)\leqq x\leqq f^{k+1}(c)$. If we now use the fact that each power of f is monotone increasing and commutes with g, we obtain

$$f^{n-1}(x) \leqq f^{n+k}(c) \leqq g(f^k(c)) \leqq g(x) \leqq g(f^{k+1}(c)) \leqq f^{n+k+2}(c) \leqq f^{n+2}(x).$$

Thus, $f^{n-1}(x) \leqq g(x) \leqq f^{n+2}(x)$, and our lemma follows.

THEOREM 3. *There exists a unique flow F on $(a, b]$ such that $F_1 = f$ and such that each F_t is continuously differentiable on $(a, b]$. Moreover, if g is any continuously differentiable homeomorphism of $(a, b]$ onto itself such that g commutes with f, then there exists a real number t such that $F_t = g$.*

PROOF. We define $F_t(x) = \Phi^{-1}(A^t\Phi(x))$ for $a < x \leqq b$. It is easy to verify that each function F_t is a homeomorphism of $(a, b]$ onto itself. Moreover, it is easy to prove that $F_sF_t = F_{s+t}$ for all s and t, and that F is continuous. Thus F is a flow on the interval $(a, b]$.

Since $\Phi'(x) = -\phi(x)$, it follows that $F_t'(x) = A^t\phi(x)/\phi(F_t(x))$ and hence F_t satisfies the differential equation $E(t)$ on the interval (a, b). Since f obviously satisfies $E(1)$, it follows from Lemma 3 that $F_1 = f$.

Since $F_1 = f$ and F is a flow, it follows that F_t commutes with f for each t. We now use Lemma 5 to obtain $\lim_{x\to b} F_t'(x) = A^t$. It follows that F_t is differentiable at b, and that $F_t'(b) = A^t$. Thus each function F_t has a continuous derivative on $(a, b]$.

Let us now prove that the flow F is unique. We observe that if n is a positive integer and g is a continuously differentiable homeomorphism of $(a, b]$ onto itself for which $g^n = f$, then

$$f'(x) = g'(g^{n-1}(x))g'(g^{n-2}(x)) \cdots g'(x).$$

If we let x tend to b, it is easy to see that we obtain

$$f'(b) = (g'(b))^n.$$

It follows that g satisfies the equation $E(1/n)$. Now let G be any flow of continuously differentiable functions which satisfies $G_1 = f$. We see that $G_{1/n}$ and $F_{1/n}$ both satisfy $E(1/n)$ for each positive integer n and hence $G_{1/n} = F_{1/n}$. It follows that $G_r = F_r$ for every rational number r. Since flows are continuous in both variables, $G_t(x) = F_t(x)$ for all real t and all x in the interval $(a, b]$. Therefore F is unique.

Now suppose that g is a continuously differentiable homeomorphism of $(a, b]$ onto itself and that g commutes with f. It follows from Corollary 1 that $g'(b) \neq 0$. Thus, since g is order preserving, $g'(b) > 0$ and there exists a real number t such that $A^t = g'(b)$. It follows from Theorem 2 that g satisfies the differential equation $E(t)$, and since F_t also satisfies $E(t)$, Lemma 3 implies that $F_t = g$.

It is obvious that we may extend the domain of f and of the functions F_t to the closed interval $[a, b]$ by defining $F_t(a) = a$, and we shall then obtain a flow F on $[a, b]$. However, as is shown by the following example, it is not necessarily true that each of the functions

F_t will have a derivative at a, even though f has a derivative at a.

EXAMPLE. Let $g(x)=4x/(3x+1)$ for $0\leqq x\leqq 1$. Next define a function f such that $f(x)=2x$ for $0\leqq x\leqq 1/3$; $f(x)=g(x)$ for $1/2\leqq x\leqq 1$; $f(x)$ arbitrary for $1/3\leqq x\leqq 1/2$, subject only to the requirement that f satisfy the conditions (i)–(v) listed at the beginning of §3. It is easily seen that it is possible to define such a function f. Theorem 3 implies that there exist flows G and F such that $G_1=g$, $F_1=f$ and, all of the functions F_t and G_t are continuously differentiable on $(0, 1]$. If one makes use of the uniqueness of G, it is easily verified that $G_t(x)=4^t x/((4^t-1)x+1)$. Since $g(x)=f(x)$ for $1/2\leqq x\leqq 1$, it is easily seen that $G_{1/2}$ and $F_{1/2}$ satisfy the same differential equation on the interval $1/2\leqq x\leqq 1$. It follows from Lemma 3 that $F_{1/2}(x)=G_{1/2}(x)$ for $1/2\leqq x\leqq 1$, and we use this fact to compute $F_{1/2}(1/2)=2/3$. Next we define points $p_0, p_1, p_2, \cdots$ by letting $p_0=1/2$, $p_1=1/3$, and $p_n=F_{-1}(p_{n-2})$ for $n\geqq 2$. Since $F_{1/2}(p_0)=2/3=F_1(p_1)$, we obtain $F_{1/2}(p_1)=p_0$. It is easy to prove by induction that $F_{1/2}(p_n)=p_{n-1}$ for every positive integer n. Since $F_{-1}(x)=x/2$ for $0\leqq x\leqq 1/2$, it is easy to compute the values of the numbers p_n, and to verify that the difference quotient $(F_{1/2}(p_n)-F_{1/2}(0))/(p_n-0)$ is equal to $3/2$ if n is odd and is equal to $4/3$ if n is even. Thus $F_{1/2}$ does not have a derivative at 0.

This example demonstrates that it is possible to have a function f which is continuously differentiable on the closed interval $[a, b]$ and which satisfies conditions (i)–(v) on the half-open interval $(a, b]$, but which cannot be embedded in a flow of functions that are continuously differentiable on the closed interval $[a, b]$.

We do, however, have the following result.

THEOREM 4. *If f is continuously differentiable over the closed interval $[a, b]$ and satisfies* (i)–(v) *on $(a, b]$, then the set G of all continuously differentiable order preserving homeomorphisms g of $[a, b]$ onto itself for which g commutes with f forms a group which is isomorphic to either the group of real numbers or the group of integers.*

PROOF. We use Theorem 3 to obtain a flow F on $[a, b]$ such that each of the functions F_t has a continuous derivative on $(a, b]$, and $F_1=f$. It follows from Theorem 3 that if $g\in G$, then there exists a real number t such that $F_t=g$. We let S be the set of all real numbers t for which $F_t\in G$. If we use the fact that $F'_{t+s}(x)=F'_t(F_s(x))F'_s(x)$ for $a<x<b$, it is easy to see that if t and s are members of S then $t+s$ is also a member of S. Moreover, if t is in S then $-t$ is in S. Thus S is a subgroup of the reals. It is not difficult to prove that G is a group which is isomorphic to S, and consequently our theorem is

proved if we can show that either S is the set R of all real numbers or else S is a nondegenerate discrete subgroup of R.

We observe that S is nondegenerate since $1 \in S$. Now assume that S is isomorphic to neither R nor the group of integers. It then follows that both S and $R-S$ are dense in R, and this implies that $R-S$ is uncountable. We shall now show that this is impossible. In order to accomplish this, we define u_t to be the lower derivate of F_t at a, and we define v_t to be the upper derivate of F_t at a. It is easily seen that if $s<t$, then $F_s(x)<F_t(x)$ for $a<x<b$, and it follows that if $s<t$ then $u_s \leqq u_t$ and $v_s \leqq v_t$. Now let r and t be members of $R-S$, $r<t$. Since S is dense in R, there exists $s \in S$ such that $r<s<t$. We see that $u_r<v_r \leqq v_s=u_s \leqq u_t<v_t$. Thus, by defining $L(t)$ to be the open interval (u_t, v_t) for each $t \in R-S$, we obtain a one-to-one function L that maps $R-S$ onto a set of mutually exclusive open intervals. This implies that $R-S$ is countable, and we have a contradiction.

We conclude by listing two unsolved problems that are of interest.

PROBLEM 1. *Find conditions on f that are necessary and sufficient for $S=R$, where S and R are the sets defined in the proof of Theorem* 4.

PROBLEM 2. *Replace condition* (v) *on f by a weaker condition.*

The solution of these problems would constitute a major step toward obtaining necessary and sufficient conditions that it be possible to embed a continuously differentiable homeomorphism (having arbitrarily many fixed points) of an interval onto itself in a flow of continuously differentiable homeomorphisms.

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