

**REMARK ON S. N. ROY'S PAPER "A USEFUL
THEOREM IN MATRIX THEORY"¹**

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The theorem in question reads as follows: If A and B are two $p \times p$ matrices, not both singular, then for all the characteristic values $c(AB)$ of the product AB we have

$$(1) \quad c_{\min}(AA^*) \cdot c_{\min}(BB^*) \leq |c(AB)|^2 \leq c_{\max}(AA^*) \cdot c_{\max}(BB^*)$$

where c_{\min} and c_{\max} stand respectively for the smallest and the largest characteristic value of AA^* , BB^* (each, of course, non-negative).

I give here a short proof of (1) which is valid also in the case when both A and B are singular.

$c = c(AB)$ being a characteristic value of AB there exists a nonzero vector x such that $ABx = cx$. Then

$$|c|^2 \|x\|^2 = \|cx\|^2 = \|ABx\|^2 \leq \|A\|^2 \|B\|^2 \|x\|^2,$$

where $\|A\|^2 = \sup_{\|y\| \leq 1} \|Ay\|^2 = \sup_{\|y\| \leq 1} (A^*Ay, y) = c_{\max}(A^*A)$, and analogously for B . Dividing by $\|x\|^2$ we get

$$(2) \quad |c(AB)|^2 \leq c_{\max}(A^*A) \cdot c_{\max}(B^*B),$$

and this is the second inequality (1), by the well-known fact that T^*T and TT^* have the same characteristic values.

The first inequality (1) is obvious if A or B is singular, for then $c_{\min}(AA^*)$ or $c_{\min}(BB^*)$ is equal to 0. If both A and B are nonsingular, then the first inequality (1) is a consequence of (2) when applied to $A_1 = B^{-1}$ and $B_1 = A^{-1}$ instead of A and B , by using the simple fact that $(T^{-1})^* = (T^*)^{-1}$ and that the characteristic values of inverse matrices are reciprocal.

The more general theorem indicated in the cited paper, i.e. that

$$\prod_{i=1}^n c_{\min}(A_i A_i^*) \leq \left| c \left(\prod_{i=1}^n A_i \right) \right|^2 \leq \prod_{i=1}^n c_{\max}(A_i A_i^*),$$

may be proved in the same manner, without supposing that at least one of the matrices A_i is nonsingular.

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² *Editorial Note*. M. F. Smiley also submitted a note which contains essentially the same proof as given in this note.