REMARK ON S. N. ROY'S PAPER "A USEFUL THEOREM IN MATRIX THEORY"[1]

BÉLA SZ.-NAGY[2]

The theorem in question reads as follows: If A and B are two $p\times p$ matrices, not both singular, then for all the characteristic values $c(AB)$ of the product AB we have

$$c_{\min}(AA^*)\cdot c_{\min}(BB^*) \leqq \left| c(AB) \right|^2 \leqq c_{\max}(AA^*)\cdot c_{\max}(BB^*) \tag{1}$$

where $c_{\min}$ and $c_{\max}$ stand respectively for the smallest and the largest characteristic value of AA^*, BB^* (each, of course, non-negative).

I give here a short proof of (1) which is valid also in the case when both A and B are singular.

$c=c(AB)$ being a characteristic value of AB there exists a nonzero vector x such that $ABx=cx$. Then

$$\left| c \right|^2 \left\| x \right\|^2 = \left\| cx \right\|^2 = \left\| ABx \right\|^2 \leqq \left\| A \right\|^2 \left\| B \right\|^2 \left\| x \right\|^2,$$

where $\left\| A \right\|^2 = \sup_{\|y\| \leq 1} \left\| Ay \right\|^2 = \sup_{\|y\| \leq 1} (A^*Ay,\ y) = c_{\max}(A^*A)$, and analogously for B. Dividing by $\left\| x \right\|^2$ we get

$$\left| c(AB) \right|^2 \leqq c_{\max}(A^*A)\cdot c_{\max}(B^*B), \tag{2}$$

and this is the second inequality (1), by the well-known fact that T^*T and TT^* have the same characteristic values.

The first inequality (1) is obvious if A or B is singular, for then $c_{\min}(AA^*)$ or $c_{\min}(BB^*)$ is equal to 0. If both A and B are nonsingular, then the first inequality (1) is a consequence of (2) when applied to $A_1=B^{-1}$ and $B_1=A^{-1}$ instead of A and B, by using the simple fact that $(T^{-1})^*=(T^*)^{-1}$ and that the characteristic values of inverse matrices are reciprocal.

The more general theorem indicated in the cited paper, i.e. that

$$\prod_{i=1}^{n} c_{\min}(A_iA_i^*) \leqq \left| c\left(\prod_{i=1}^{n} A_i \right) \right|^2 \leqq \prod_{i=1}^{n} c_{\max}(A_iA_i^*),$$

may be proved in the same manner, without supposing that at least one of the matrices A_1 is nonsingular.

UNIVERSITY OF SZEGED, HUNGARY

Received by the editors February 1, 1955.

[1] Proc. Amer. Math. Soc. vol. 5 (1954) pp. 635–638.

[2] *Editorial Note.* M. F. Smiley also submitted a note which contains essentially the same proof as given in this note.