A TAUBERIAN THEOREM FOR α-CONVERGENCE OF CESÀRO MEANS

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The object of this note is to generalize certain Tauberian results proved by Gehring [3] for summability $(C, k; \alpha)$. The notation is as in [3], with the following additional definitions: If $k > -1$, then A_n^k, B_n^k denote the nth Cesàro sums of order k for the series $\sum_{n=0}^{\infty} a_n$, $\sum_{n=0}^{\infty} b_n$, where $b_n = na_n$. A_n^{-1}, B_n^{-1} denote a_n, b_n. Summability $(C, -1; \alpha)$ of $\sum a_n$ will be taken to mean summability $(C, 0; \alpha)$ of $\sum a_n$ together with the condition that sequence $\{na_n\}$ be α-convergent to 0.

Gehring's Tauberian theorems are:

THEOREM 4.3.2. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $\sum a_n$ *is summable* $(A; \alpha)$ *to* S. *If the sequence* $\{na_n\}$ *is* α*-convergent to* 0, $\sum a_n$ *is* α*-convergent to* S.

THEOREM 4.3.3. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $\sum a_n$ *is summable* $(A; \alpha)$ *to* S. *Then* $\sum a_n$ *is* α*-convergent to* S *if and only if the sequence* $\{(a_1 + \cdots + na_n)/n\}$ *is* α*-convergent to* 0.

THEOREM 4.3.4. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $\sum a_n$ *is* α*-convergent. If the sequence* $\{na_n\}$ *is* α*-convergent to* 0, $\sum a_n$ *is summable* $(C, k; \alpha)$ *to its sum for every* $k > -1$.

These will be used in the proof of the following:

THEOREM. *Suppose that* $0 \leqq \alpha \leqq 1$ *and that* $\sum a_n$ *is summable* $(A; \alpha)$ *to* S. *Then, for* $r \geqq -1$, $\sum a_n$ *is summable* $(C, r; \alpha)$ *to* S *if and only if the sequence* $\{B_n^r / C_n^{r+1}\}$ *is* α*-convergent to* 0.

PROOF OF THEOREM. *Necessity.* If $r = -1$ this follows immediately from the definition of summability $(C, -1; \alpha)$. If $r > -1$ then by the consistency theorem for $(C, r; \alpha)$ summability (Gehring [3, Theorem 4.2.1]) it follows that both sequences $\{S_n^r\}$, $\{S_n^{r+1}\}$ are α-convergent to S. By Hardy [1, Equation (6.1.6)],

$$S_n^r = S_n^{r+1} + \frac{1}{r+1} \frac{B_n^r}{C_n^{r+1}} \tag{1}$$

and the result follows since a linear combination of sequences summable $(C, k; \alpha)$ is itself summable $(C, k; \alpha)$.

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Sufficiency. If $r>-1$ it may be shown[1] as in Szász [4, §1], that

$$\frac{1}{y+1}\sum_{n=0}^{\infty} S_n^{r+1}\left(1-\frac{1}{y+1}\right)^n = \frac{r+1}{y}\int_0^y\left(1-\frac{u}{y}\right)^r\phi(u)du, \tag{2}$$

where $\phi(u)=f(1-1/(u+1))$ and $f(x)=\sum a_n x^n$.

CASE (i). $\alpha=0$, $r>-1$. Since $\sum a_n$ is summable (A) to S it follows that $\phi(u)$ tends to S as u increases. The right-hand side of (2), being the $(r+1)$th transform of $\phi(u)$, also tends to S; and so the sequence $\{S_n^{r+1}\}$ is summable $(A;\alpha)$ to S.

CASE (ii). $0<\alpha\leqq 1$, $r>-1$. Putting

$$g(y)=\frac{1}{y+1}\sum_{n=0}^{\infty} S_n^{r+1}\left(1-\frac{1}{y+1}\right)^n,$$

we get, from (2), that

$$g(y)=(r+1)\int_0^1(1-v)^r\phi(vy)dv,$$

where $\phi(u)$ now has bounded α-variation over $(0, \infty)$. Let

$$\begin{aligned} V &= \left[\sum_{\nu=1}^{N} \left| g(y_\nu)-g(y_{\nu-1})\right|^{1/\alpha}\right]^\alpha \\ &= (r+1)\left[\sum_{\nu=1}^{N}\left|\int_0^1(1-v)^r\{\phi(vy_\nu)-\phi(vy_{\nu-1})\}dv\right|^{1/\alpha}\right]^\alpha. \end{aligned}$$

Then by Theorem 201 of [5],[2]

$$\begin{aligned} V &\leqq (r+1)\int_0^1(1-v)^r\left\{\sum_{\nu=1}^{N}\left|\phi(vy_\nu)-\phi(vy_{\nu-1})\right|^{1/\alpha}\right\}^\alpha dv \\ &\leqq (r+1)M\int_0^1(1-v)^r dv \\ &= M, \end{aligned}$$

where $M=V_\alpha\{\phi(x);\, 0\leqq x<\infty\}$. Thus $g(y)$ has bounded α-variation over $(0, \infty)$ and so the series $\sum s_n$, where $s_n=S_n^{r+1}-S_{n-1}^{r+1}$, is summable $(A;\alpha)$ to S. Further, by Hardy [1, Equation (6.1.6)],

$$ns_n = n(S_n^{r+1}-S_{n-1}^{r+1}) = B_n^r/C_n^{r+1},$$

[1] Note however the error in Szász's equation (2.4). There, and in the previous line, occurs an extraneous term $(1+1/y)^{\alpha-1}$.

[2] I am indebted to a referee for shortening my argument at this step.

so that sequence $\{ns_n\}$ is α-convergent to 0. Hence by Theorem 4.3.2 we have that $\sum s_n$ and sequence $\{S_n^{r+1}\}$ are α-convergent to S.

It is readily seen from Minkowski's inequality that the sum of two α-convergent sequences is also α-convergent, and we therefore deduce from (1) that $\{S_n^r\}$ is α-convergent to S; i.e., $\sum a_n$ is summable $(C, r; \alpha)$ to S.

CASE (iii). $r=-1$. When $\alpha=0$ the result reduces to Tauber's original theorem; when $0<\alpha<1$ it follows from Theorem 4.3.2. For $\alpha=1$ the result was proved by Hyslop [**2**, Theorem 4].

REFERENCES

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