

ON SOME PROPERTIES OF THE NORMAL AND GAMMA DISTRIBUTIONS

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It is well-known [2] that in a normal population any translation-invariant statistic is stochastically independent of the sample mean. Similarly [3] in a gamma population the sample mean is distributed independently of any scale-invariant statistic.

In the present note we shall prove the following theorems.

THEOREM 1. *Let x_1, x_2, \dots, x_n be n identically and independently distributed normal variables. The necessary and sufficient condition that the sum $x_1 + x_2 + \dots + x_n$ is distributed independently of some function $g(x_1, x_2, \dots, x_n)$ is that $g(x_1, x_2, \dots, x_n)$ and $g(x_1 + \lambda, x_2 + \lambda, \dots, x_n + \lambda)$ should be identically distributed.*

PROOF OF NECESSITY. From the conditions of Theorem 1, it follows that

$$(1) \quad E\left(\exp\left(it \sum x + iug\right)\right) = E\left(\exp\left(it \sum x\right)\right) \cdot E(\exp(iug))$$

where $i = (-1)^{1/2}$ as usual and t and u are real.

Without any loss of generality, we may assume x 's to be distributed normally with zero mean and unit variance.

Then (1) gives

$$(2) \quad \begin{aligned} & E\{\exp(iug(x_1, x_2, \dots, x_n))\} \\ &= \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(iug(x_1, x_2, \dots, x_n)\right) \\ & \quad - \frac{1}{2} \sum (x - it)^2 dx_1 \dots dx_n = \phi(t, u), \text{ say.} \end{aligned}$$

Now from (2) it follows that $\phi(t, u)$ does not involve t for all real values of t .

But it is evident that $\phi(t, u)$ is an entire function in t and hence it should be free of t for all complex values of t also.

Hence putting $t = -i\lambda$ (λ any real number) we have, after some simplifications,

$$(3) \quad \phi(-i\lambda, u) = E\{\exp(iug(x_1 + \lambda, \dots, x_n + \lambda))\}.$$

Thus combining (2) and (3) we have

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$$(4) \quad E\{\exp(iug(x_1, x_2, \dots, x_n))\} \\ = E\{\exp(iug(x_1 + \lambda, x_2 + \lambda, \dots, x_n + \lambda))\}.$$

Then using the uniqueness theorem of characteristic functions [1], it follows from (4) that $g(x_1, x_2, \dots, x_n)$ and $g(x_1 + \lambda, x_2 + \lambda, \dots, x_n + \lambda)$ should be identically distributed.

PROOF OF SUFFICIENCY. As before let us assume x 's to be distributed normally with zero mean and unit variance.

Then the characteristic function of the joint cumulative distribution of $\sum x$ and $g(x_1, x_2, \dots, x_n)$ is given by

$$(5) \quad E\left(\exp(iu \sum x + iug)\right) \\ = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(iu \sum x + iug(x_1, \dots, x_n) - \frac{1}{2} \sum x^2\right) dx_1 \dots dx_n \\ = E\left(\exp(iu \sum x)\right) \cdot \phi(t, u), \text{ say,}$$

where

$$\phi(t, u) = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(iug(x_1, \dots, x_n) - \frac{1}{2} \sum (x - it)^2\right) dx_1 \dots dx_n.$$

Now it is given that $g(x_1, x_2, \dots, x_n)$ and $g(x_1 + \lambda, x_2 + \lambda, \dots, x_n + \lambda)$ are identically distributed and hence their corresponding characteristic functions should be identical, that is,

$$(6) \quad E\{\exp(iug(x_1, x_2, \dots, x_n))\} \\ = E\{\exp(iug(x_1 + \lambda, x_2 + \lambda, \dots, x_n + \lambda))\} \\ = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(iug(x_1 + \lambda, \dots, x_n + \lambda) - \frac{1}{2} \sum x^2\right) dx_1 \dots dx_n \\ = \frac{1}{(2\pi)^{n/2}} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \exp\left(iug(x_1, \dots, x_n) - \frac{1}{2} \sum (x - \lambda)^2\right) dx_1 \dots dx_n \\ = \phi(-i\lambda, u).$$

Thus it follows from (6) that $\phi(t, u)$ which is defined only for real t and real u in (5) must also exist for all complex t , since then it represents the characteristic function of the distribution of $g(x_1, x_2, \dots, x_n)$ under the assumptions of the theorem.

Again from (6) it is obvious that $\phi(t, u)$ does not involve t for all complex t . Hence $\phi(t, u)$ being an entire function in t should be free of t for all real t also.

Thus we have the relation

$$(7) \quad E\left(\exp(it \sum x + iug)\right) = E\left(\exp(it \sum x)\right) \cdot E(\exp(iug)).$$

Then the stochastic independence of $x_1 + x_2 + \dots + x_n$ and $g(x_1, x_2, \dots, x_n)$ follows immediately from (7).

Proceeding exactly in the same way we can prove a corresponding theorem for the case of gamma variates which may be stated as follows:

THEOREM 2. *Let x_1, x_2, \dots, x_n be n independently and identically distributed gamma variates with the distribution function*

$$dF(x) = \frac{1}{\Gamma(p)} e^{-x} x^{p-1} dx \quad (0 \leq x \leq \infty).$$

Then the necessary and sufficient condition that the sum $\sum x$ is distributed independently of some function $g(x_1, \dots, x_n)$ is that $g(x_1, \dots, x_n)$ and $g(\lambda x_1, \dots, \lambda x_n)$ should be identically distributed.

In conclusion, the author expresses his thanks to the referee for some valuable comments.

Note added in proof: It is easy to construct examples to show that for the case of independent observations from a normal (or gamma) distribution, some statistic may be distributed independently of the mean without being translation- (or scale-) invariant. A number of such examples have been given by the author in his doctoral thesis *On characterizations of probability distributions and statistics*, submitted to Calcutta University in 1955.

REFERENCES

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