QUASI-EQUICONTINUOUS SETS OF FUNCTIONS[1]

CHIEN WENJEN

The well-known theorem of analysis that if F is a family of functions defined, equicontinuous, and uniformly bounded on a bounded closed set E in n-dimensional real cartesian space R^n, then from every sequence $\{f_n\}$ of functions of F it is possible to select a uniformly convergent subsequence, has been recently generalized to various abstract spaces [**1**; **4**; **6**]. Consider a set F of continuous functions on one topological space X to another, Y. For any point x of X and any open set W of Y we denote by (x, W) the totality of functions f in F for which $f(x) \in W$. The topology in F obtained by employing all sets of the (x, W) as a subbase in F is called the p-topology by Arens [**2**]. The purpose of this note is to find the necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of continuous functions and the necessary and sufficient conditions that a set of continuous functions be compact in the p-topology.

DEFINITION. Let $\{f_n\}$ be a sequence of functions from a topological space X to be metric space Y. $\{f_n\}$ is said to be ϵ-related at a point $x \in X$ if for every arbitrarily chosen $\epsilon > 0$ there is a neighborhood $U(x)$ of x such that, corresponding to each point $x' \in U(x)$, a positive number $N_\epsilon(x, x')$ can be determined satisfying the condition:

$$\rho[f_n(x), f_n(x')] < \epsilon$$

whenever $n > N_\epsilon(x, x')$.

DEFINITION. Let F be a family of continuous functions from a topological space X to a metric space Y. F is called quasi-equicontinuous if in every infinite subset Q of F and at any point $x \in X$ there is a sequence $\{f_n\}$ contained in Q which is ϵ-related at x.

THEOREM. *If X is locally separable and Y metric, a set of functions $F \subset Y^X$, where Y^X denotes the set of all continuous functions from X to Y, is compact under p-topology if and only if*

(1) *F is closed in Y^X,*

(2) *$F(x) = \bigcup_{f \in F} f(x)$ is compact for every $x \in X$,*

(3) *F is quasi-equicontinuous.*

Presented to the Society, February 26, 1955; received by the editors December 31, 1954 and, in revised form, April 11, 1955.

[1] This problem was suggested by Professor Richard Arens to whom the writer is indebted for many valuable suggestions.

PROOF. I. *Necessity.* (1) F is closed since F is compact and Y^X is a Hausdorff space under p-topology.

(2) Let $g_x(f)=f(x)$. Then g_x is a continuous function of f and the compactness of $F(x)$ follows from the compactness of F.

(3) Since compactness implies countable compactness, any infinite subset Q of F has a limit point f contained in F. Let $\{x_n\}$ be an enumerable set contained and dense in a neighborhood $U(x_0)$ of a point x_0 in X. We can find a subset $\{f_n\}$ of Q satisfying

$$\begin{aligned}&\rho[f_n(x_0), f(x_0)] < 1/n,\\ &\rho[f_n(x_1), f(x_1)] < 1/n,\\ &\cdots\cdots\cdots,\\ &\rho[f_n(x_n), f(x_n)] < 1/n, \qquad n = 1, 2, 3, \cdots.\end{aligned}$$

Then

$$f_n(x_k) \rightarrow f(x_k), \qquad k = 1, 2, 3, \cdots$$

as $n \rightarrow \infty$.

Next we show that $f_n(x) \rightarrow f(x)$ at any point x in $U(x_0)$. Suppose on the contrary that $f_n(x)$ does not converge to $f(x)$ at certain point x' in $U(x_0)$. There exist an $\epsilon>0$ and a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that

$$\text{(A)} \qquad \rho[f_{n_i}(x'), f(x')] > \epsilon, \qquad i = 1, 2, 3, \cdots.$$

Let g be a limit point of $\{f_{n_i}\}$ in F and let a subsequence $\{f_{n'_i}\}$ of $\{f_{n_i}\}$ be so chosen that

$$\begin{aligned}&\rho[f_{n'_i}(x'), g(x')] < 1/i,\\ &\rho[f_{n'_i}(x_1), g(x_1)] < 1/i,\\ &\cdots\cdots\cdots,\\ &\rho[f_{n_i}(x_i), g(x_i)] < 1/i, \qquad i = 1, 2, 3, \cdots.\end{aligned}$$

Then

$$\begin{aligned}&f_{n'_i}(x') \rightarrow g(x'),\\ &f_{n'_i}(x_k) \rightarrow g(x_k), \qquad k = 1, 2, 3, \cdots,\end{aligned}$$

as n'_i approaches to infinity. Now it is clear that

$$\lim f_{n'_i}(x_k) = \lim f_{n_i}(x_k) = f(x_k) = g(x_k), \qquad k = 1, 2, 3, \cdots.$$

We have therefore

$$f(x) = g(x)$$

for all x in $U(x_0)$ on account of the continuity of the functions $f(x)$ and $g(x)$. It follows that

$$f_{n'_i}(x') \to g(x') = f(x'). \tag{B}$$

The contradiction between the relations (A) and (B) proves that $f_n(x)$ converges to $f(x)$ at any point x in $U(x_0)$. In other words,

$$\lim_{n\to\infty} \lim_{x\to x_0} f_n(x) = \lim_{x\to x_0} \lim_{n\to\infty} f_n(x).$$

Hence $\{f_n\}$ is ϵ-related at x_0.[2] The quasi-equicontinuity of the set of functions F is proved.

II. *Sufficiency*. Since $F(x)$ is compact for any $x \in X$, the topological product $G = \prod_{x\in X} F(x)$ is compact. Consider the correspondence between F and a subset S of G obtained by assigning to each $f \in F$ the point in G with coordinates $f(x)$, x ranging over X; this correspondence is a homeomorphism. In order to prove the compactness of F it is sufficient to prove that S is compact, that is, to prove that S is closed in G.

Let g be a limit point of S with coordinates $g(x)$. There exists a sequence $\{f_n\} \subset F$ such that

$$\begin{aligned} &\rho[f_n(x_0), g(x_0)] < 1/n, \\ &\rho[f_n(x_1), g(x_1)] < 1/n, \\ &\cdots\cdots\cdots\cdots, \\ &\rho[f_n(x_n), g(x_n)] < 1/n, \qquad n = 1, 2, 3, \cdots, \end{aligned}$$

where $\{x_n\}$ is an enumerable set dense in a neighborhood $U(x_0)$ of x_0. By the quasi-equicontinuity of the set of functions F there is a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that for each $\epsilon > 0$ there is a neighborhood $V(x_0)$ of x_0 contained in $U(x_0)$ such that

$$\rho[f_{n_i}(x_0), f_{n_i}(x)] < \epsilon$$

for any $x \in V(x_0)$ and all $n_i > N_\epsilon(x_0, x)$. Now

$$\begin{aligned} \rho[g(x_0), g(x_k)] < \rho[g(x_0), f_{n_i}(x_0) &+ \rho[f_{n_i}(x_0), f_{n_i}(x_k)] \\ &+ \rho[f_{n_i}(x_k), g(x_k)] < 3\epsilon \end{aligned}$$

for any $x_k \in V(x_0)$, if n_i is sufficiently large. By the same reasoning for any point x in $V(x_0)$ there is a neighborhood $W(x)$ of x contained in $U(x_0)$ such that

[2] That the ϵ-related condition was given by Hobson as a necessary and sufficient condition for interchange of order in repeated limits was pointed out by the referee [6, p. 409].

$$\rho[g(x), g(x_k)] < 3\epsilon \quad \text{if} \quad x_k \in \{x_n\} \cap W(x).$$

Then

$$\rho[g(x_0), g(x)] < \rho[g(x_0), g(x_j)] + \rho[g(x_j), g(x)] < 6\epsilon$$

for any $x \in V(x_0)$, where $x_j \in V(x_0) \cap W(x)$. $g(x)$ is therefore continuous at x_0, that is, g belongs to S and the closedness of S is proved.

COROLLARY. *Let F be a family of continuous functions from a separable space X to a metric space Y. The necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of functions of F are*:

(1) *$F(x)$ is countably compact for any $x \in X$,*

(2) *F is quasi-equicontinuous.*

COROLLARY. *Let (C) be the set of all continuous functions defined on the closed interval* (0, 1) *and let F be any subset of (C). F is weakly compact if and only if it is weakly closed and quasi-equicontinuous.*

REFERENCES

1. R. Arens, *A topology for spaces of transformations*, Ann. of Math. vol. 47 (1946) pp. 480–495.

2. ———, *Duality in linear spaces*, Duke Math. J. vol. 14 (1947) pp. 787–794.

3. E. F. Eberlein, *Weak compactness in Banach space* I, Proc. Nat. Acad. Sci. U.S.A. vol. 33 (1947) pp. 51–53.

4. D. Gale, *Compact sets of functions and function rings*, Proc. Amer. Math. Soc. vol. 1 (1950) pp. 303–308.

5. E. W. Hobson, *The theory of functions of a real variable and the theory of Fourier's series*, vol. 1, 3d ed.

6. S. B. Myers, *Equicontinuous sets of mappings*, Ann. of Math. (2) vol. 47 (1946) pp. 496–502.

UNIVERSITY OF CALIFORNIA, LOS ANGELES