QUASI-EQUICONTINUOUS SETS OF FUNCTIONS

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The well-known theorem of analysis that if \( F \) is a family of functions defined, equicontinuous, and uniformly bounded on a bounded closed set \( E \) in \( n \)-dimensional real cartesian space \( \mathbb{R}^n \), then from every sequence \( \{f_n\} \) of functions of \( F \) it is possible to select a uniformly convergent subsequence, has been recently generalized to various abstract spaces \([1; 4; 6]\). Consider a set \( F \) of continuous functions on one topological space \( X \) to another, \( Y \). For any point \( x \) of \( X \) and any open set \( W \) of \( Y \) we denote by \((x, W)\) the totality of functions \( f \) in \( F \) for which \( f(x) \in W \). The topology in \( F \) obtained by employing all sets of the \((x, W)\) as a subbase in \( F \) is called the \( \rho \)-topology by Arens [2]. The purpose of this note is to find the necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of continuous functions and the necessary and sufficient conditions that a set of continuous functions be compact in the \( \rho \)-topology.

**Definition.** Let \( \{f_n\} \) be a sequence of functions from a topological space \( X \) to a metric space \( Y \). \( \{f_n\} \) is said to be \( \epsilon \)-related at a point \( x \in X \) if for every arbitrarily chosen \( \epsilon > 0 \) there is a neighborhood \( U(x) \) of \( x \) such that, corresponding to each point \( x' \in U(x) \), a positive number \( N_{\epsilon}(x, x') \) can be determined satisfying the condition:

\[
\rho[f_n(x), f_n(x')] < \epsilon
\]

everywhere \( n > N_{\epsilon}(x, x') \).

**Definition.** Let \( F \) be a family of continuous functions from a topological space \( X \) to a metric space \( Y \). \( F \) is called quasi-equicontinuous if in every infinite subset \( Q \) of \( F \) and at any point \( x \in X \) there is a sequence \( \{f_n\} \) contained in \( Q \) which is \( \epsilon \)-related at \( x \).

**Theorem.** If \( X \) is locally separable and \( Y \) metric, a set of functions \( F \subset Y^X \), where \( Y^X \) denotes the set of all continuous functions from \( X \) to \( Y \), is compact under \( \rho \)-topology if and only if

1. \( F \) is closed in \( Y^X \),
2. \( F(x) = \bigcup_{f \in F} f(x) \) is compact for every \( x \in X \),
3. \( F \) is quasi-equicontinuous.

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Proof. I. Necessity. (1) $F$ is closed since $F$ is compact and $Y^x$ is a Hausdorff space under $\rho$-topology.

   (2) Let $g_x(f) = f(x)$. Then $g_x$ is a continuous function of $f$ and the compactness of $F(x)$ follows from the compactness of $F$.

   (3) Since compactness implies countable compactness, any infinite subset $Q$ of $F$ has a limit point $f$ contained in $F$. Let $\{x_n\}$ be an enumerable set contained and dense in a neighborhood $U(x_0)$ of a point $x_0$ in $X$. We can find a subset $\{f_n\}$ of $Q$ satisfying

   \[ \rho[f_n(x_0), f(x_0)] < 1/n, \]
   \[ \rho[f_n(x_1), f(x_1)] < 1/n, \]
   \[ \ldots \ldots \ldots \ldots \ldots \ldots \]
   \[ \rho[f_n(x_n), f(x_n)] < 1/n, \quad n = 1, 2, 3, \ldots. \]

   Then

   \[ f_n(x_k) \to f(x_k), \quad k = 1, 2, 3, \ldots \]

   as $n \to \infty$.

   Next we show that $f_n(x) \to f(x)$ at any point $x$ in $U(x_0)$. Suppose on the contrary that $f_n(x)$ does not converge to $f(x)$ at certain point $x'$ in $U(x_0)$. There exist an $\epsilon > 0$ and a subsequence $\{f_{n_i}\}$ of $\{f_n\}$ such that

   \[ \rho[f_{n_i}(x'), f(x')] > \epsilon, \quad i = 1, 2, 3, \ldots. \]

   Let $g$ be a limit point of $\{f_{n_i}\}$ in $F$ and let a subsequence $\{f_{n_i}'\}$ of $\{f_{n_i}\}$ be so chosen that

   \[ \rho[f_{n_i}'(x'), g(x')] < 1/i, \]
   \[ \rho[f_{n_i}'(x_i), g(x_i)] < 1/i, \]
   \[ \ldots \ldots \ldots \ldots \ldots \ldots \]
   \[ \rho[f_{n_i}(x_i), g(x_i)] < 1/i, \quad i = 1, 2, 3, \ldots. \]

   Then

   \[ f_{n_i}'(x') \to g(x'), \]
   \[ f_{n_i}'(x_k) \to g(x_k), \quad k = 1, 2, 3, \ldots, \]

   as $n_i'$ approaches to infinity. Now it is clear that

   \[ \lim f_{n_i}'(x_k) = \lim f_{n_i}(x_k) = f(x_k) = g(x_k), \quad k = 1, 2, 3, \ldots. \]

   We have therefore

   \[ f(x) = g(x) \]
for all \( x \) in \( U(x_0) \) on account of the continuity of the functions \( f(x) \) and \( g(x) \). It follows that
\[
(B) \quad f_n(x') \to g(x') = f(x').
\]
The contradiction between the relations (A) and (B) proves that \( f_n(x) \) converges to \( f(x) \) at any point \( x \) in \( U(x_0) \). In other words,
\[
\lim_{n \to \infty} \lim_{x \to x_0} f_n(x) = \lim_{x \to x_0} \lim_{n \to \infty} f_n(x).
\]
Hence \( \{f_n\} \) is \( \epsilon \)-related at \( x_0 \). The quasi-equicontinuity of the set of functions \( F \) is proved.

II. Sufficiency. Since \( F(x) \) is compact for any \( x \in X \), the topological product \( G = \prod_{x \in X} F(x) \) is compact. Consider the correspondence between \( F \) and a subset \( S \) of \( G \) obtained by assigning to each \( f \in F \) the point in \( G \) with coordinates \( f(x) \), \( x \) ranging over \( X \); this correspondence is a homeomorphism. In order to prove the compactness of \( F \) it is sufficient to prove that \( S \) is compact, that is, to prove that \( S \) is closed in \( G \).

Let \( g \) be a limit point of \( S \) with coordinates \( g(x) \). There exists a sequence \( \{f_n\} \subset F \) such that
\[
\rho[\rho[f_{n_0}(x_0), g(x_0)] < 1/n,
\rho[\rho[f_{n_1}(x_1), g(x_1)] < 1/n,
\ldots 
\rho[\rho[f_{n_n}(x_n), g(x_n)] < 1/n, \quad n = 1, 2, 3, \ldots ,
\]
where \( \{x_n\} \) is an enumerable set dense in a neighborhood \( U(x_0) \) of \( x_0 \). By the quasi-equicontinuity of the set of functions \( F \) there is a subsequence \( \{f_{n_i}\} \) of \( \{f_n\} \) such that for each \( \epsilon > 0 \) there is a neighborhood \( V(x_0) \) of \( x_0 \) contained in \( U(x_0) \) such that
\[
\rho[ho[f_{n_i}(x_0), f_{n_i}(x)] < \epsilon
\]
for any \( x \in V(x_0) \) and all \( n_i > N_i(x_0, x) \). Now
\[
\rho[\rho[g(x_0), g(x_k)] < \rho[\rho[g(x_0), f_{n_i}(x_0)] + \rho[\rho[f_{n_i}(x_0), f_{n_i}(x_k)]
\rho[ho[f_{n_i}(x_k), g(x_k)] < 3\epsilon
\]
for any \( x_k \in V(x_0) \), if \( n_i \) is sufficiently large. By the same reasoning for any point \( x \) in \( V(x_0) \) there is a neighborhood \( W(x) \) of \( x \) contained in \( U(x_0) \) such that

\[\text{That the } \epsilon \text{-related condition was given by Hobson as a necessary and sufficient condition for interchange of order in repeated limits was pointed out by the referee [6, p. 409].}\]
\[ \rho [g(x), g(x_k)] < 3\varepsilon \quad \text{if} \quad x_k \in \{ x_n \} \cap W(x). \]

Then
\[ \rho [g(x_0), g(x)] < \rho [g(x_0), g(x_j)] + \rho [g(x_j), g(x)] < 6\varepsilon \]
for any \( x \in V(x_0) \), where \( x_j \in V(x_0) \cap W(x) \). \( g(x) \) is therefore continuous at \( x_0 \), that is, \( g \) belongs to \( S \) and the closedness of \( S \) is proved.

**Corollary.** Let \( F \) be a family of continuous functions from a separable space \( X \) to a metric space \( Y \). The necessary and sufficient conditions that it be possible to select a subsequence converging pointwise to a continuous function from any given sequence of functions of \( F \) are:

1. \( F(x) \) is countably compact for any \( x \in X \),
2. \( F \) is quasi-equicontinuous.

**Corollary.** Let \( (C) \) be the set of all continuous functions defined on the closed interval \((0, 1)\) and let \( F \) be any subset of \((C)\). \( F \) is weakly compact if and only if it is weakly closed and quasi-equicontinuous.

**References**


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