THE NUMBER OF LATTICE POINTS IN AN n-DIMENSIONAL TETRAHEDRON[1]

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Let the a_i $(1 \leqq i \leqq n)$ be positive integers, no two of which have a common factor; let A denote the product of the a_i.

In this paper we prove the following two exceedingly simple theorems,[2] which we have not, to our surprise, seen before.

THEOREM 1. *If* $\eta \equiv 0 \pmod{A}$, *the number of solutions of*

$$\sum_{i=1}^{n} a_i x_i \leqq \eta, \qquad x_i \geqq 0 \qquad (1 \leqq i \leqq n) \tag{1}$$

is a polynomial with rational coefficients in the quantities η/A *and the* a_i.

THEOREM 2. *If* $\eta \equiv 0 \pmod{A}$, *the number of solutions of*

$$\sum_{i=1}^{n} a_i x_i = \eta, \quad \text{each } x_i \geqq 0, \tag{2}$$

is a polynomial with rational coefficients in the quantities η/A *and the* a_i.

One should mention here that both theorems are completely trivial for $n=1$. Theorem 2 is also trivial for $n=2$, and indeed for the following reason. Clearly if $\eta \equiv 0 \pmod{a_1 a_2}$,

$$a_1 x_1 + a_2 x_2 = \eta$$

implies that $x_1 \equiv 0 \pmod{a_2}$, $x_2 \equiv 0 \pmod{a_1}$. Let $x_1 = a_2 x_1'$, $x_2 = a_1 x_2'$. Hence we see that if $\eta \equiv 0 \pmod{a_1 a_2}$ the number of solutions of

$$a_1 x_1 + a_2 x_2 = \eta, \qquad x_1 \geqq 0,\ x_2 \geqq 0 \tag{3}$$

is the same as the number of solutions of

$$x_1 + x_2' = \frac{\eta}{a_1 a_2}, \qquad x_1' \geqq 0,\ x_2' \geqq 0 \tag{4}$$

which is clearly $(\eta/a_1 a_2) + 1$.

However, Theorem 2, while not deep, does not seem to be trivial for $n=3$.

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[2] For the case when the a_i's are all equal to 1, the result is well known.

EXAMPLE. For $n=3$ (in Theorem 2) and $\eta\equiv 0 \pmod{a_1a_2a_3}$, the number of solutions of $\sum_{i=1}^{3} a_ix_i=\eta$ is

$$\frac{\eta^2}{2a_1a_2a_3}+\frac{\eta(a_1+a_2+a_3)}{2a_1a_2a_3}+1.$$

Theorems 1 and 2 are proved as follows. Following D. H. Lehmer [1] and D. C. Spencer [2] we write $L(\eta)$ for the number of solutions of (1), when η is any non-negative real number; we use $r(\eta)$ to denote the number of solutions of (2). Observe that $r(\eta)=0$ if η is not an integer. We also write

$$L^*(\eta) = L(\eta) - r(\eta)/2. \tag{5}$$

We have (see [2])

$$L^*(\eta) = \frac{1}{2\pi i}\int \frac{e^{\eta S}dS}{S\prod_{i=1}^{n}(1 - e^{-a_iS})} \tag{6}$$

where the integral is along the vertical line $\sigma=c$ ($c>0$; $S=\sigma+it$, σ and t real). Now $L^*(\eta)$ is a polynomial in η of degree n, where the coefficients B_i ($1\leqq i\leqq n$) of η^i, multiplied by A, are polynomials in a_i. The constant term B_0, which is a function of the a_i and η, *is periodic in* η, with a period equal to A. (See G. H. Hardy [3] for the special case $n=2$.)

From the periodic nature of B_0 it is trivial that when $\eta\equiv 0 \pmod{A}$, $B_0=B_0(\eta;\, a_1, \cdots, a_n)=B_0(0;\, a_1, \cdots, a_n)$. The B_i are found as follows:

Our function $f(S)$ has:

(i) A pole of order $n+1$ at $S=0$. The residue $P(\eta)$ here is a polynomial in η with coefficients which, when multiplied by A, are polynomials (with rational coefficients) in the a_i. This remark is true for all non-negative real η.

(ii) A pole of order n at $S=2m\pi i$ (m an integer $\neq 0$). Here again, the residue $Q(\eta)$ is a polynomial in η with coefficients which, when multiplied by A, are polynomials (with rational coefficients) of the a_i *provided* η *is an integer.*

(iii) Simple poles at $S=2g_i\pi i/a_i$ where $g_i\not\equiv 0 \pmod{a_i}$. The residue here is denoted by $T_i(\eta)$, and has a period a_i. (The argument in Hardy [3] generalizes at once to prove this statement.) $T_i(\eta)$ is an infinite trigonometrical series. (Incidentally, in another note we show how to reduce these to finite trigonometrical sums.) Thus[3]

[3] In fact for all $\eta\geqq 0$, $T_i(\eta)=\sum_{g_i\not\equiv 0 \pmod{a_i}} \exp(\eta(2g_i\pi i/a_i))/2g_i\pi i\cdot H_i$, where $H_i=\prod_{u\neq i}(1-\exp((-2g_i\pi i a_u/a_i))$.

$$(7) \qquad L^*(\eta) = P(\eta) + Q(\eta) + \sum_{i=1}^{n} T_i(\eta) \, [\eta \text{ an integer} \geqq 0].$$

Finally we evaluate $r(\eta)$ when $\eta \equiv 0 \pmod{A}$ as follows. The latter condition on η holds throughout the rest of the paper, and we shall not repeat it. Similar to our proof of (7) we easily establish that

$$(8) \quad L^*(\eta + 1/2) = P(\eta + 1/2) + R(\eta + 1/2) + \sum_{i=1}^{n} T_i(\eta + 1/2).$$

The R function here is not the same as the Q introduced before. However, it remains, like Q, a polynomial in η with coefficients which, when multiplied by A, are polynomials (with rational coefficients) in the a_i. Observe that

$$(9) \qquad \begin{aligned} \sum_{i=1}^{n} T_i\left(\eta + \frac{1}{2}\right) &= \sum_{i=1}^{n} T_i\left(\frac{1}{2}\right) \\ &= L^*\left(\frac{1}{2}\right) - P\left(\frac{1}{2}\right) - R\left(\frac{1}{2}\right) \\ &= 1 - P\left(\frac{1}{2}\right) - R\left(\frac{1}{2}\right) \end{aligned}$$

and

$$(10) \qquad r(\eta) = 2\{L^*(\eta + 1/2) - L^*(\eta)\}.$$

From the above equations we see that Theorems 1 and 2 follow. In fact $r(\eta)$ is a polynomial in η of degree $n-1$ with coefficients which, when multiplied by A, are polynomials (with rational coefficients) in the a_i. An identical remark is true about $L(\eta)$, except that we have, in this case, a polynomial of degree n in η.

References

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3. G. H. Hardy, *Ramanujan*, Cambridge University Press, 1940.

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