

A HYPERBOLIC SURFACE IN 3-SPACE¹

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In a talk at the 1954 International Congress,² the author outlined an existence proof for a surface of the form $z=f(x, y)$ which covers every point of the x, y -plane exactly once, and which is conformally equivalent to the interior of the unit circle.³ This paper gives a simple method for the explicit construction of such a surface. It is based on the following observations regarding the type of a simply-connected Riemann surface.

LEMMA. If a parabolic Riemann surface R containing a line of symmetry L is mapped conformally onto the complex w -plane, then the image of L is a straight line.

NOTE. By a line of symmetry we mean a straight or polygonal line about which the surface is symmetric in some embedding.

PROOF. If we map one half of the surface conformally onto the upper half of the z -plane, then L must correspond to the whole real axis, since otherwise the surface would be hyperbolic. The symmetrical map onto the lower half-plane will also be conformal, and by the Schwarz reflection principle, the map will remain conformal along L . If L is polygonal, the vertices can be at most removable singularities. We thus obtain a map of R onto the entire z -plane, with L corresponding to the real axis. The composed map of the z -plane onto the w -plane must then be linear, and the image of the real axis is a straight line.

COROLLARY. If a Riemann surface contains a pair of intersecting lines of symmetry and a third line of symmetry disjoint from both of these, then it is hyperbolic.

PROOF. If it were parabolic the image of the intersecting lines of symmetry would be a pair of intersecting straight lines in the plane,

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² *On a conjecture in the problem of type for simply-connected Riemann surfaces*, Proceedings of the International Congress of Mathematicians, Amsterdam, 1954, vol. II, p. 153.

³ We may note that the question which our construction settles was originally raised by Loewner, and was communicated to the Princeton conference on Riemann surfaces by Professor Bers.

and the third line of symmetry would have to correspond to a straight line disjoint from both of these, which is impossible.

We now construct a surface which is symmetrical about the x and y axes, and also about the polygonal line L , indicated in Fig. 1. The symmetry about L will not be apparent in the initial construction, but can be seen from a re-embedding of the surface. To make the procedure clear, we give first the idea of the construction, and then carry out the details.

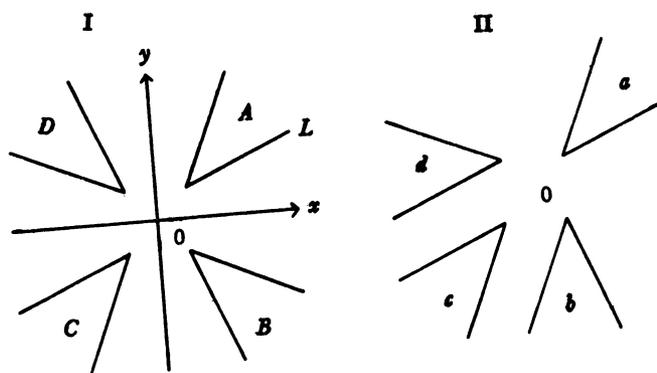


FIG. 1

We designate by I the portion of the x, y -plane which one obtains by removing the four angular sectors, $A, B, C,$ and D of Fig. 1. L is the boundary of sector A . We replace sector A by a piece of surface I' which is essentially the "reflection" of I over L . We replace $B, C,$ and D by congruent pieces, using symmetry with respect to the x and y axes. The missing sectors of I' which correspond to $B, C,$ and D , are now replaced by "reflections" over L of the surface pieces just placed over $B, C,$ and D , and these in turn are reflected in the x and y axes. Continuing this process inductively, we arrive at a surface which is by its very construction symmetric about the x and y axes and L . The only difficulty is in finding a method of "reflecting" over L . This is done by making use of "fan-shaped" surfaces. For precision we introduce the following terminology.

DEFINITION. A *blade* of angle α is a closed plane sector spanned by two rays which make an acute angle α between them. We shall assume throughout this paper that one of the two rays is parallel to the x, y -plane, and that the plane of the blade is not vertical.

A *fan* is the union of an even number of blades, each obtained by reflection in the vertical plane through the terminal ray of the pre-

vious one. The initial ray of the first blade will be assumed to be parallel to the x, y -plane.

We shall allow the special case where all the blades lie in a single plane parallel to the x, y -plane. In this case the fan will itself be a plane sector.

It follows immediately from the definition that every fan covers in a one-one manner a sector of the x, y -plane. Furthermore we have the following fundamental property: given an arbitrary n -bladed fan with blades of angle α , and given an arbitrary sector in the x, y -plane of angle $\beta \leq n\alpha$, we can find an isometric image of the fan which projects onto the given sector. Namely, we place one edge of the first blade along one of the boundary rays of the given sector and then rotate the blade until it projects on a subsector of angle β/n . For this the blade must make an angle γ with the x, y -plane, where

$$\gamma = \cos^{-1} \frac{\tan(\beta/n)}{\tan \alpha}.$$

The fan generated by this blade will have the required properties. This process of re-embedding we shall refer to as "placing a fan over a sector."

In the following we shall work with fans from which three sectors have been removed. The surface is constructed by successively adjoining such fans starting with I. If I were itself a fan, then by placing a copy of it over sector A we could carry out the process outlined above without change. Since it is not, we require two steps to obtain actual symmetry about L .

The exact construction is as follows. To obtain I we draw radial slits to infinity from the four points $(\pm 1, \pm 1)$, and remove the 45° angular sectors symmetric about each of these slits. If we replace the radial slits from $(1, -1)$ and $(-1, 1)$ by vertical and horizontal slits respectively, and again remove symmetric 45° sectors, we obtain a portion of the plane denoted by II in Fig. 1. III is obtained from II by displacing the removed sector at $(1, 1)$ through an angle of 45° in the negative direction, and IV is the reflection of III in the line $x=y$. If sectors b, c , and d are replaced we may consider II, III, and IV as fans with vertex at $(1, 1)$ with 14 blades of angle $\pi/8$. The blades are indicated in Fig. 2 where the heavier dotted lines are those which remain parallel to the x, y -plane when the fan is placed over a sector. Since the sides of b, c , and d are parallel to these lines, they will remain parallel to the x, y -plane, so that when these sectors are removed they may be replaced by fans.

The final surface S is defined as the limit of approximating surfaces S_n , each one obtained from the previous one by adjoining a finite number of fans. S_1 is obtained by placing a copy of II over each of the four removed sectors of I. To obtain S_2 from S_1 we adjoin three new fans to the copy of II over sector A : we replace b by a copy of III, c by II, and d by IV. The crucial point in the construction is that

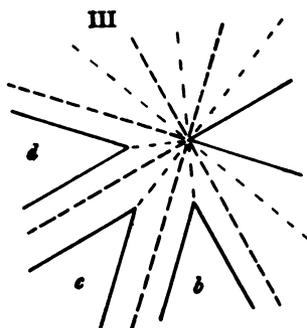


FIG. 2

the surface thus obtained is divided into two congruent parts by L . More properly, we should say that the two parts are isometric, because they cannot be superimposed by a rigid motion in space, but only if we allow bending. Namely, the two parts are identical except for the portions lying over $B \cup b$ and $D \cup d$. But in both cases we see that the portion lying over $B \cup b$ (going clockwise around $(1, -1)$) consists of an isometric copy of II followed by two blades of angle $\pi/8$, while the portion over $D \cup d$ is just the reflection of this in the line $x = y$.

The S_n are now defined by induction. We shall assume that S_{2n-1} is symmetric about the x and y axes, and that S_{2n} is obtained from S_{2n-1} by adjoining fans over sector A in such a manner that S_{2n} is divided by L into two congruent parts, S'_{2n} and S''_{2n} , where S'_{2n} lies over sector A . S_{2n+1} is then obtained by reflecting the fans just adjoined over the x and y axes. Their images will be fans placed over certain sectors of S''_{2n} . We obtain S_{2n+2} by placing a copy of each of these fans over the corresponding sectors of S'_{2n} .

From this construction it follows immediately that S_{2n+1} is symmetric about the x and y axes, and S_{2n+2} is divided into two congruent parts by L .

We now observe that given an arbitrary circle about the origin of the x, y -plane, there exists N such that S_N covers this circle, and

hence the corresponding part of S_n remains unchanged for $n \geq N$. Thus $S = \lim S_n$ is well defined and covers the whole x, y -plane. Furthermore, we have also $S = \lim S_{2n} = \lim S_{2n+1}$. Since each S_{2n} is symmetric about the x and y axes, so is S . To re-embed S so that L is a line of symmetry we need only rotate the edges of S'_2 till they form a straight angle. L is then a straight line and reflecting S'_2 over it gives a congruent copy of S'_2 and hence of S''_2 . The union of these two is therefore congruent to S_2 , and adjoining the same fans as before to the surface in this embedding makes each S_{2n+1} symmetric about L , and hence S in this embedding is also symmetric about L .

REMARK 1. The surface S is not strictly a Riemann surface because the vertices of the removed sectors will be singular points. However, one can easily show that the lemma and corollary at the beginning of this paper also hold for such surfaces.

REMARK 2. The construction given here is not the same as that indicated by the author at the International Congress. The latter will be discussed as part of a longer paper on the Euclidean embedding of Riemann surfaces.

REMARK 3. By changing S in a neighborhood of every singular point, one can obtain a hyperbolic surface without singularities. One can in fact, as will be shown in the above-mentioned paper, construct a surface which is everywhere infinitely differentiable.

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