ON A MIXED BOUNDARY VALUE PROBLEM OF HARMONIC FUNCTIONS[1]

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The mixed boundary value problem has been treated recently by several authors [**4**; **5**; **6**; **7**]. In this note we give an existence proof using subharmonic functions.

Consider a 2-dimensional multiply connected domain D with the boundary Γ, which consists of k closed curves $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$. The boundary curves Γ_i $(i=1, 2, \cdots, k)$ are assumed to have continuous tangents.

A real-valued continuous function $f(\zeta)$ is defined on a part α of the boundary. (The boundary points are denoted by ζ.) α consists of a finite number of disjoint closed curves or arcs α_i. If α_i is an arc, then we denote by P_i', P_i'' its two end points. $|f(\zeta)| \leqq M$.

The remaining part of Γ we denote by β. Another real valued continuous function $g(\zeta)$ is defined on the closure of β.

The problem is to determine a function $u(z)$ in D, such that:

(a) $u(z)$ is harmonic in D,

(b) $\lim_{z\to\zeta} u(z) = f(\zeta)$ on α_i and $\limsup_{z\to\zeta} |u(z)| \leqq M$ at $\zeta = P_i', P_i''$,

(c) $\lim_{z\to\zeta} u(z)$ is continuous for ζ on β,

(d) $\partial u/\partial n = g(\zeta)$ for ζ on β, where the differentiation is along the inward normal n.

We shall treat first the problem by assuming $g(\zeta)=0$.[2] Since Neumann's problem can be solved for smooth boundary, the assumption $g(\zeta)=0$ on β is not a restriction.

Define the class $\mathcal{J}$ of admissible functions $v(z)$ with the properties:

(a) $v(z)$ is continuous subharmonic in D,

(b) $\limsup_{z\to\zeta} v(z) \leqq f(\zeta)$ for ζ on α,

(c) $\lim_{z\to\zeta} v(z)$ exists and is continuous for ζ on β,

(d) $\liminf \Delta v/\Delta n \geqq 0$ on β. The lim inf is taken along the normal pointing into the interior of D.

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[2] The same problem was treated by R. Courant by the method of variational calculus [**2**, p. 40].

First, the class $\mathcal{J}$ is not empty, because all the constants $\leqq -M$ belong to it. Second, due to the property d, the functions in $\mathcal{J}$ have M as an upper bound. To prove that, we make use of a theorem by E. Hopf [3], according to which any admissible subharmonic function v of class C'' does not attain its maximum on β. If v is only continuous but not necessarily differentiable, Hopf's theorem remains valid for v. For, Hopf's proof goes through also in this case if one replaces his condition $L(v)\geqq 0$ by that of continuous subharmonic functions v, and applies the maximum principle. Hence we have $v\leqq M$ in $D+\beta$, for all v in $\mathcal{J}$.

The harmonic function $u(z)$ will be determined by a modification of the well known method of O. Perron [9] on the Dirichlet problem.

We define the function u at each point z in D by

$$u(z) = \text{l.u.b. } v(z), \tag{1}$$

where the l.u.b. is taken over all v in $\mathcal{J}$. This definition is justified because the functions v are bounded from above.

Lemma 1. *The function $u(z)$ is harmonic in D.*

Proof. Suppose $v_1, v_2 \in \mathcal{J}$. We shall show that the function $V(z) = \max(v_1, v_2)$ also belongs to $\mathcal{J}$. It is easily seen that the function $V(z)$ has the properties a and b. Consider a point $P(\zeta)$ on β. Since v_1 and v_2 are continuous at P, we have $\lim_{z\to\zeta} V(z) = \lim \max_{z\to\zeta}(v_1, v_2) = \lim_{z\to\zeta}[(v_1+v_2)+|v_1-v_2|]/2$, and hence V is continuous at P. Let $Q(z)$ be a point on the inward normal at $P(\zeta)$, and let $\Delta v_i = v_i(Q) - v_i(P)$, $i=1, 2$. It is easily seen that $\Delta V \geqq \min(\Delta v_1, \Delta v_2)$. From this follows $\liminf \Delta V/\Delta n \geqq 0$.

What is said for V is also valid for $\max(v_1, v_2, \cdots, v_n) = V_n$. We may now carry out Perron's construction in the customary manner [1, p. 197] by forming a maximizing sequence $\{V_n\}$ of subharmonic functions in an arbitrary disk Δ_1, whose closure is contained in D. The functions V_n' which are equal to V_n outside and on the boundary of Δ_1, and equal to the Poisson integrals in Δ_1, form a nondecreasing sequence which converges to a harmonic function in Δ_1. This limit function is equal to $u(z)$. Since Δ_1 is an arbitrary disk, the function u is harmonic in D.

Lemma 2. *The function $u(z)$ determined by* (1) *satisfies* $\lim_{z\to\zeta} u(z) = f(\zeta)$ *on α, except possibly the end points.*

Proof. To prove the lemma we have to show that $\limsup_{z\to\zeta_0} u(z) \leqq f(\zeta_0)+\epsilon$ and $\liminf_{z\to\zeta_0} u(z) \geqq f(\zeta_0)-\epsilon$ for all $\epsilon>0$ and ζ_0 on α. The first inequality can be proved in the same manner as it is known for the Dirichlet problem [1, p. 198].

To prove the second inequality consider a simply connected subdomain Δ contained in D, such that a part δ of its boundary is on α. The remaining part of the boundary of Δ we denote by γ. We can solve the Dirichlet problem in Δ with the boundary function $F=f(\zeta)-\epsilon$, $\epsilon>0$, on δ and $F=v_0$ on γ, where v_0 are the values on γ of an admissible function v. We obtain a harmonic function $H(z)$ in Δ.

Consider the function $W=H$ in Δ and $W=v$ in the rest of D. W is subharmonic in D, $\lim_{z\to\zeta} W \leqq f(\zeta)$ on α, $W=v$ on β, therefore W is in $\mathfrak{J}$.

Now $W=f(\zeta_0)-\epsilon$ at ζ_0. As an admissible function $W(z)\leqq u(z)$ and $\liminf_{z\to\zeta_0} u(z) \geqq W(\zeta_0)=f(\zeta_0)-\epsilon$. Hence $\lim_{z\to\zeta} u(z)=f(\zeta)$ for all points of α, except P_i', P_i''.

LEMMA 3. *$u(z)$ is continuous and $\partial u/\partial n=0$ on β.*

PROOF. We shall make use of conformal mapping.[3] Consider a simply connected subdomain Δ' contained in D. The boundary of Δ' is a simple closed curve. It consists of two arcs, one denoted by δ' is a closed subarc of β, while the other, denoted by γ', is an open arc which lies in D. We can map Δ' conformally onto a semicircular domain d so that γ' goes into the circular arc and δ' into the bounding diameter. Denote this mapping by S.

Consider the maximizing sequence $\{V_n\}$, which was used for construction of u. Each V_n has continuous boundary values h_n on γ'. By S the function h_n goes into a continuous function ϕ_n on the circular arc of d. With the boundary function ϕ_n and its symmetric extension on the other semicircle we obtain a harmonic function H_n in the closed disk. H_n is symmetric with respect to the diameter and has a vanishing normal derivative on the diameter.

Now we transform H_n back to Δ' by S^{-1}, and get a harmonic function G_n, which has a vanishing normal derivative on δ'. $G_n=V_n$ on γ'. Since

$$\liminf \left(\frac{\Delta V_n}{\Delta n}-\frac{\Delta G_n}{\Delta n}\right) \geqq \liminf \frac{\Delta V_n}{\Delta n} - \limsup \frac{\partial G_n}{\partial n} \geqq 0 \tag{2}$$

on δ', we conclude that $G_n \geqq V_n$ in Δ' (by making use again of Hopf's lemma [3]). The function equal to G_n in Δ' and to V_n outside of Δ' is an admissible function. Hence the functions G_n form a maximizing sequence $\{G_n\}$, which converges to u. Hence u is continuous and $\partial u/\partial n=0$ on δ', consequently on β.

[3] This is the only step in the proof that can not be used to treat the case of more than 2 independent variables.

LEMMA 4. *The function* $|u(z)| \leqq M$ *in* $D+\Gamma$.

PROOF. From Lemmas 2 and 3 by the boundary behavior of u it follows that $u \leqq M$ in $D+\Gamma$. u may not be defined at the end points P_i', P_i''. Since $u \geqq -M$ we have $|u| \leqq M$.

Uniqueness of the solution. Suppose that there exists a harmonic function $u_1(z)$ which solves the problem and is different from $u(z)$. Consider the function $U = u - u_1$. U is harmonic in D, $\lim_{z\to\zeta} U = 0$ on the curves and open arcs α_i, U is continuous on β and bounded at the points P_i', P_i'' and furthermore $\partial U/\partial n = 0$ on β. We shall show that the maximum of U is on α. Suppose that the supremum of U is M_1 and is equal to the supremum at a point P which is one of the P_i', P_i''. Consider a small circle r around P, such that no other P_i', P_i'' are in it. The part of β inside the circle r is denoted by β_r, the intersection of D and r by d_r. Let $m = \max U$ on d_r, $m < M_1$. By reflecting d_r on β_r (through a coñformal mapping as in Lemma 3) we obtain a domain in which the harmonic function is bounded by the constant m or zero, except at the point P. We can apply the extended maximum principle for harmonic functions [**8**, p. 115] to conclude that the supremum is not at P. The maximum is not on β [**3**]. Therefore it is on α and $U \leqq 0$ in $D+\Gamma$. By considering the function $-U$ instead of U, we get $-U \leqq 0$. Both inequalities together imply $U = 0$ or $u \equiv u_1$.

BIBLIOGRAPHY

1. L. Ahlfors, *Complex analysis*, New York, McGraw-Hill, 1953.

2. R. Courant, *Dirichlet's principle, conformal mapping and minimal surfaces*, New York, Interscience, 1950.

3. E. Hopf, *A remark on linear elliptic differential equations of second order*, Proc. Amer. Math. Soc. vol. 3 (1952).

4. J. H. Huth, *Mixed boundary value problems in potential theory*, Journal of the Franklin Institute vol. 257 (1954) pp. 121–124.

5. Y. Komatu, *Eine gemischte Randwertaufgabe für einen Kreis*, Proc. Japan Acad. vol. 28 (1952) pp. 339–341.

6. ———, *Mixed boundary value problems*, Journal of the Faculty of Science. Imperial University of Tokyo. Section I vol. 6 (1953) pp. 345–391.

7. Y. Komatu and Imsik Hong, *On mixed boundary value problems*, Kodai Math. Sem. Rep. (1953) pp. 65–76.

8. R. Nevanlinna, *Uniformisierung*, Berlin, Springer, 1953.

9. O. Perron, *Eine neue Behandlung der ersten Randwertaufgabe für* $\Delta u = 0$, Math. Zeit. vol. 18 (1923) pp. 42–54.

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