INCLUSION RELATIONS AMONG SOME METHODS OF SUMMABILITY

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1. Introduction. This paper deals with inclusion relations among three matrix methods of summability, the Euler methods, $E(p)$, the Taylor methods, $T(r)$, and certain methods related to the Taylor methods and designated $S(q)$. Matrices which define these methods are

$$E(p) \equiv E_{nk}(p) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad p \neq 0; \tag{1}$$

$$T(r) \equiv T_{nk}(r) = (1-r)^{n+1} \binom{k}{n} r^{k-n}, \qquad r \neq 1; \tag{2}$$

$$S(q) \equiv S_{nk}(q) = (1-q)^{n+1} \binom{n+k}{k} q^k, \quad q \neq 0, q \neq 1. \tag{3}$$

Particular attention is paid to the case when the orders p, q, and r are complex constants.

Necessary and sufficient conditions for the regularity of the $E(p)$ and $T(r)$ methods are, respectively, $0 < p \leqq 1$ [**3**], and $0 \leqq r < 1$ [**4**]. $S(q)$ is regular if $0 < q < 1$ [**2**], and by means of the Silverman-Toeplitz theorem it is easily shown that $S(q)$ is regular only if $0 < q < 1$.

Letting DW denote the domain of values of z for which the geometric series is summable to $(1-z)^{-1}$ by the method W, we have

$$DE(p): \; |z - (1 - 1/p)| < |1/p| \; [\mathbf{3}], \tag{4}$$

$$DT(r): \; |z| < |1/r|, \qquad |z - rz| < |1 - rz| \; [\mathbf{5}], \tag{5}$$

$$DS(q): \; |z| < |1/q|, \qquad |z - 1/q| > |1 - 1/q|. \tag{6}$$

This last domain is readily obtained after transforming the sequence $s_k = (1 - z^{k+1})/(1-z)$ by the matrix $S_{nk}(q)$.

Agnew [**3**] has dealt with the inclusion $E(p_2) \supset E(p_1)$ for p_1 and p_2 complex, and Laush [**4**] has similarly treated $T(r_2) \supset T(r_1)$.

2. $S(q_2) \supset S(q_1)$. Meyer-König [**2**] has shown that for $0 < q < 1$ one does not obtain inclusion relations among the $S(q)$ methods. A similar result for the complex case is contained in

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THEOREM 1. *If* $|q_1|<1$, $|q_2|<1$, *and* $q_1 \neq q_2$, *there are no values of* q_1 *and* q_2 *for which* $S(q_2) \supset S(q_1)$.

PROOF. Let C_1, C_2, C_3, and C_4 denote respectively the four circles with equations

$$\begin{aligned} |z| &= |1/q_1|, \\ |z - 1/q_1| &= |1 - 1/q_1|, \\ |z| &= |1/q_2|, \\ |z - 1/q_2| &= |1 - 1/q_2|. \end{aligned}$$

Now $S(q_2) \not\supset S(q_1)$ if there exist points which lie in $DS(q_1)$ but not in $DS(q_2)$. This situation will occur if there exist points which lie inside C_1, C_3, and C_4, but outside C_2. Let R denote this region. Because $DS(q_2)$ lies inside C_3, there is no hope of having $S(q_2) \supset S(q_1)$ unless $|1/q_2| > |1/q_1|$, so hereafter we make that assumption. If $1/q_1$, $1/q_2$, and 1 are not collinear, C_2 and C_4 both pass through $z=1$ but are not tangent to each other. It is easily seen that there exist points arbitrarily close to $z=1$ which belong to R. If $1/q_1$, $1/q_2$, and 1 are collinear, let L denote the line segment through these three points. Now C_2 and C_4 are tangent at $z=1$. If $1/q_1$ and $1/q_2$ lie on the same side of 1 there exist points close to 1 which are inside C_4 and outside C_2 and so belong to R. If $1/q_1$ and $1/q_2$ lie on opposite sides of 1, one encounters points of R when moving along L from 1 toward $1/q_2$. Thus, in all cases there are points which belong to R, so $S(q_2) \not\supset S(q_1)$.

3. **Matrix products.** In the remainder of our inclusion considerations we shall use the results of matrix products to suggest relationships between the orders p, q, and r that may constitute necessary and sufficient conditions for inclusion.

Because $E^{-1}(p) = E(1/p)$ we consider first the product $S(q)E(1/p)$.

THEOREM 2. *If* $|p|<1$, $|q|<1$, *and* $|q(1-1/p)|<1$, *then*

$$S(q)E(1/p) = S\left(\frac{q}{p+q-pq}\right).$$

PROOF. For non-negative values of n, m, and k,

$$\binom{n+k}{k}\binom{k}{m} = \binom{n+k}{n+m}\binom{n+m}{m},$$

so

$$
\begin{aligned}
S_{nk}(q)E_{km}(1/p) &= \sum_{k=0}^{\infty}(1-q)^{n+1}\binom{n+k}{k}q^k\binom{k}{m}(1/p)^m(1-1/p)^{k-m}\\
&= \frac{(1-q)^{n+1}}{p^m(1-1/p)^m}\binom{n+m}{m}\sum_{k=m}^{\infty}\binom{n+k}{n+m}(q-q/p)^k\\
&= \frac{(1-q)^{n+1}q^m}{p^m}\binom{n+m}{m}\sum_{i=0}^{\infty}\binom{n+m+i}{i}(q-q/p)^i\\
&= (1-q)^{n+1}(q/p)^m\binom{n+m}{m}1/(1-q+q/p)^{n+m+1},
\end{aligned}
$$

because $|q(1-1/p)|<1$. Finally,

$$
\begin{aligned}
S_{nk}(q)E_{km}(1/p) &= \left[\frac{p(1-q)}{p+q-pq}\right]^{n+1}\binom{n+m}{m}\left[\frac{q}{p+q-pq}\right]^m\\
&= S_{nm}\left(\frac{q}{p+q-pq}\right).
\end{aligned}
$$

4. **Inferences from the matrix calculations.** Theorem 2 suggests conditions under which $S(q)$ might include $E(p)$ when use is made of the following formal matrix calculation:

If $S(q)E^{-1}(p)\supset I$, then right multiplying by $E(p)$ and assuming associativity yields $S(q)\supset E(p)$. But $S(q/[p+q-pq])$ is regular if and only if

$$
0<\frac{q}{p+q-pq}<1 \text{ or } 1=\left|\frac{q}{p+q-pq}\right|+\left|1-\frac{q}{p+q-pq}\right|,
$$

which is equivalent to

$$
|1/p+1/q-1| = |1/p| + |1/q-1|. \tag{7}
$$

This implies the collinearity of the points $1/q$, 1, and $1-1/p$ with the point $z=1$ lying between the other two points.

With p and q related by (7), upon comparing $DS(q)$ and $DE(p)$ we encounter another relationship between p and q which, in conjunction with (7), might guarantee inclusion of $E(p)$ by $S(q)$, namely

$$
|1/q| > |1/p| + |1-1/p|. \tag{8}
$$

Conditions (7) and (8) turn out to be sufficient for the desired inclusion.

5. $S(q)\supset E(p)$.

THEOREM 3. *If* $|p|<1$, $|q|<1$, *and if* p *and* q *satisfy conditions* (7) *and* (8), *then* $S(q)\supset E(p)$.

PROOF. Suppose that a given sequence $\{s_i\}$ is summable $E(p)$ to s. The $E(p)$ transform of $\{s_i\}$ is given by

$$\sigma_k = \sum_{i=0}^{k} \binom{k}{i} p^i(1-p)^{k-i}s_i \quad \text{and} \quad \lim_k \sigma_k = s.$$

Writing $\{s_n\}$ as the $E(1/p)$ transform of $\{\sigma_k\}$ and taking the $S(q)$ transform of $\{s_n\}$ yields

$$(9) \qquad t_m = (1-q)^{m+1} \sum_{n=0}^{\infty} \binom{m+n}{n} q^n \sum_{k=0}^{n} \binom{n}{k} (1/p)^k(1-1/p)^{n-k}\sigma_k.$$

The convergence of $\{\sigma_k\}$ implies that $|\sigma_k| \leqq M$ for $k=0, 1, 2, \cdots$. Then

$$\begin{aligned} |t_m| &\leqq M|1-q|^{m+1} \sum_{n=0}^{\infty} \binom{m+n}{n} |q|^n \sum_{k=0}^{n} \binom{n}{k} |1/p|^k |1-1/p|^{n-k} \\ &= M|1-q|^{m+1} \sum_{n=0}^{\infty} \binom{m+n}{n} |q|^n (|1/p| + |1-1/p|)^n \\ &= \frac{M|1-q|^{m+1}}{[1-|q|(|1/p|+|1-1/p|)]^{m+1}} \end{aligned}$$

where the last step is justified by (8). Since the series in (9) converge absolutely, it is permissible to invert the order of summation and write

$$\begin{aligned} t_m &= (1-q)^{m+1} \sum_{k=0}^{\infty} (1/p)^k(1-1/p)^{-k}\sigma_k \sum_{n=k}^{\infty} \binom{m+n}{n}\binom{n}{k}(q-q/p)^n \\ &= (1-q)^{m+1} \sum_{k=0}^{\infty} \binom{m+k}{k} (1/p)^k(1-1/p)^{-k}\sigma_k \\ &\quad \cdot \sum_{i=0}^{\infty} \binom{m+i+k}{m+k} (q-q/p)^{i+k} \\ &= (1-q)^{m+1} \sum_{k=0}^{\infty} \binom{m+k}{k} (q/p)^k\sigma_k \frac{1}{(1-q+q/p)^{m+k+1}} \\ &= \left(1 - \frac{q}{p+q-pq}\right)^{m+1} \sum_{k=0}^{\infty} \binom{m+k}{k} \left(\frac{q}{p+q-pq}\right)^k \sigma_k, \end{aligned}$$

where $|q-q/p| < 1$ because of (8). Now $\{t_m\}$ is the $S(q/[p+q-pq])$ transform of $\{\sigma_k\}$, and the regularity of this matrix, which is assured by (7), is sufficient in order that $\{s_i\}$ be summable $S(q)$ to s, and that $S(q) \supset E(p)$.

COROLLARY. *If* $0<p<1$, $0<q<1$, *and* $1+1/q>2/p$, *then* $S(q) \supset E(p)$.

THEOREM 4. *If* $|p|<1$, $|q|<1$, *and* $S(q)\supset E(p)$, *then the following conditions hold*:

$$|1/p+1/q-1| = |1/p| + |1/q-1| \tag{7}$$

$$|1/q| \geqq |1/p| + |1-1/p|. \tag{10}$$

PROOF. Let C_1 denote the circle $|z-(1-1/p)|=|1/p|$, and C_2 denote the circle $|z-1/q|=|1-1/q|$. If (7) does not hold, the centers of C_1 and C_2 are nor collinear with 1. Then it is readily seen that $DE(p)$ and $DS(q)$ overlap, and so (7) is necessary. Now suppose that (7) holds, but that $|1/q|<|1/p|+|1-1/p|$. In this case a point z can be chosen which lies inside C_1 but outside $|z|=|1/q|$. This implies that $z\in DE(p)$ but $z\notin DS(q)$, hence $S(q)\not\supset E(p)$, and so (10) is necessary.

COROLLARY. *If* $0<p<1$, $0<q<1$, *and* $S(q)\supset E(p)$, *then* $1+1/q \geqq 2/p$.

In comparing Theorem 3 and its corollary with Theorem 4 and its corollary we note the unresolved problem suggested by the presence of the equality sign in (10) and in the last corollary.

6. $E(p)\supset T(r)$. Using the same approach to study the relation $E(p)\supset T(r)$ we first obtain the matrix product

$$E(p)T(p) = S(p), \tag{11}$$

which follows after a straightforward calculation and use of the binomial coefficient relation

$$\sum_{n=0}^{m}\binom{k}{n}\binom{m}{m-n} = \binom{k+m}{m}.$$

Since $T^{-1}(r)=T(r/[r-1])$, $E(p)T^{-1}(r)=S(p)$ if $p=r/(r-1)$. If $0<p<1$, $S(p)$ is regular, and possibly $E(p)\supset T(r)$. That these conditions do turn out to be sufficient is brought out by

THEOREM 5. *If* $p=r/(r-1)$ *and* $0<p<1$, *then* $E(p)\supset T(r)$.

PROOF. Suppose that a given sequence $\{s_i\}$ is summable $T(r)$ to s. Let $\{\sigma_k\}$ denote the $T(r)$ transform of $\{s_i\}$. Since $T^{-1}(r)=T(p)$, the $E(p)$ transform of the $T^{-1}(r)$ transform of $\{\sigma_k\}$ is given by

$$t_n = \sum_{m=0}^{n}\binom{n}{m}p^m(1-p)^{n-m}(1-p)^{m+1}\sum_{k=m}^{\infty}\binom{k}{m}p^{k-m}\sigma_k. \tag{12}$$

Because the summation on m in (12) is finite we may invert the order of summation, obtaining

$$\begin{aligned} t_n &= \sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{m}\binom{k}{m}p^k(1-p)^{n+1}\sigma_k \\ &\quad + \sum_{k=n+1}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\binom{k}{m}p^k(1-p)^{n+1}\sigma_k \\ &= \sum_{k=0}^{\infty}\binom{n+k}{k}(1-p)^{n+1}p^k\sigma_k. \end{aligned}$$

Thus $\{t_n\}$ is the $S(p)$ transform of $\{\sigma_k\}$, and so when $0<p<1$, $S(p)$ is regular, $\{s_i\}$ is summable $E(p)$ to s, and $E(p)\supset T(r)$.

A condition which is both necessary and sufficient for this inclusion is contained in

THEOREM 6. *If* $p=r/(r-1)$ *and* $0<|r|<1$, *then* $E(p)\supset T(r)$ *if and only if* $0<p<1/2$.

PROOF. The sufficiency of the condition $0<p<1/2$ follows from Theorem 5. To prove necessity, suppose that p does not lie between 0 and 1/2. Then, since $0<|r|<1$, it follows that $p<0$ and $0<r<1$. For any $p<0$, $1-1/p>1$ and so $DE(p)$ lies entirely to the right of the point $z=1$, and thus does not contain the point $z_0=0$. For any r for which $0<r<1$, $DT(r)$ contains z_0. Hence $z_0\in DT(r)$ but $z_0\notin DE(p)$ and $E(p)$ cannot include $T(r)$.

7. $S(q)\supset T(r)$. We employ the same technique in considering $S(q)\supset T(r)$. Using the relation

$$\sum_{k=0}^{n}\binom{m+k}{k}\binom{n}{k}(-1)^k = (-1)^n\binom{m}{n}$$

we obtain the matrix product $S(q)T^{-1}(q)=S(q)T(q/[q-1])=E(q)$. This result is in harmony with (11) but does not follow from it. In this case, the inference is that if $0<q<1$, $S(q)\supset T(q)$. However, as we shall see in Theorem 7, we need to have

$$\left|\frac{|q|}{|1-q|-|q|}\right| < 1. \tag{13}$$

This condition, when taken with $0<q<1$, leads to the restriction $0<q<1/3$.

THEOREM 7. *If* $0<q<1/3$, *then* $S(q)\supset T(q)$.

PROOF. If $0<q<1/3$, then

$$\left|\frac{q}{1-q}\right| < \frac{1}{2}. \tag{14}$$

Suppose that a given sequence $\{s_i\}$ is summable $T(q)$ to s. Let $\{\sigma_k\}$ denote the $T(q)$ transform of $\{s_i\}$. The $S(q)$ transform of the $T^{-1}(q)$ transform of $\{\sigma_k\}$ is given by

$$\begin{aligned} t_m = (1-q)^{m+1} \sum_{n=0}^{\infty} \binom{m+n}{n} q^n \frac{1}{(1-q)^{n+1}} \\ \cdot \sum_{k=n}^{\infty} \binom{k}{n} (-1)^{k-n} \left(\frac{q}{1-q}\right)^{k-n} \sigma_k. \end{aligned} \tag{15}$$

Since $\{\sigma_k\}$ is bounded, absolute convergence of the series in (15) can be shown by using (14) and (13). Then

$$\begin{aligned} t_m &= \sum_{n=0}^{\infty} \sum_{k=n}^{\infty} \binom{m+n}{n} \binom{k}{n} q^k (1-q)^{m-k} (-1)^{k-n} \sigma_k \\ &= \sum_{k=0}^{\infty} (-q)^k (1-q)^{m-k} \sigma_k \sum_{n=0}^{k} \binom{m+n}{n} \binom{k}{n} (-1)^n \\ &= \sum_{k=0}^{\infty} (-q)^k (1-q)^{m-k} \sigma_k (-1)^k \binom{m}{k} \\ &= \sum_{k=0}^{m} \binom{m}{k} q^k (1-q)^{m-k} \sigma_k. \end{aligned}$$

Thus $\{t_m\}$ is the $E(q)$ transform of $\{\sigma_k\}$, and so when $0<q<1/3$, $E(q)$ is regular, $\{s_i\}$ is summable $S(q)$ to s, and $S(q) \supset T(q)$.

8. **Conclusion.** Owing to analytic difficulties encountered in computing the matrix product $T(r)E^{-1}(p)$, and our inability to obtain $S^{-1}(q)$, the matrix approach to the study of the relations $T(r) \supset E(p)$, $E(p) \supset S(q)$, and $T(r) \supset S(q)$ gave no results.

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