ON THE OPTIMUM GRADIENT METHOD FOR SYSTEMS OF LINEAR EQUATIONS

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In the optimum gradient procedure for solving the system of linear equations

(1) $$Ax = b$$

the choice of metric is arbitrary. In this note we consider a comparison of metrics induced by certain matrix functions of A in terms of rapidity of convergence. A is an n-square, nonsingular, complex matrix, x and b are complex, column n-vectors, z^* is the conjugate transpose of z, $\lambda(A)$ denotes an arbitrary eigenvalue of A, and $\lambda_m(A)$ and $\lambda_M(A)$ are the minimum and maximum eigenvalues of A respectively when all $\lambda(A)$ are real. In this case we denote by I_A the closed interval $[\lambda_m(A), \lambda_M(A)]$.

If R is a positive-definite Hermitian (p.d.h.) matrix define

(2) $$\|z\|_R^2 = z^*Rz.$$

Solving (1) is then equivalent to minimizing the function

(3) $$\phi(x) = \|Ax - b\|_R^2.$$

Denote by $L(x, d)$ the set of vectors $x-\alpha d$ for $\alpha\in(-\infty, \infty)$. In [2] the following relations are established:

(4) $$\min_{x\in L(x_0, d)} \phi(x) = \phi(x_0 - \alpha_0 d)$$

where $\alpha_0 = \mathrm{Re}(d^*(Bx_0-c)/d^*Bd)$, $B=A^*RA$, $c=A^*Rb$; and

(5) $$\phi(x_0) - \phi(x_0 - \alpha_0 d) = \alpha_0^2 d^*Bd.$$

α_0 is called the *optimum* α in direction d at x_0. Note that B is p.d.h. In [2] the *optimum gradient* procedure for solving (1) is defined as follows:

(i) let x_0 be arbitrary,

(ii) set $x_{k+1} = x_k - \alpha_k z_k$ where $z_k = Bx_k - c$ and α_k is the optimum α in direction z_k at x_k. From (5) we see that

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$$\phi(x_{k+1}) = \phi(x_k) - \alpha_k^2 z_k^* B z_k$$

and hence

$$\frac{\phi(x_{k+1})}{\phi(x_k)} = 1 - \alpha_k \frac{z_k^* z_k}{\phi(x_k)}$$

(6)
$$= 1 - \frac{z_k^* z_k}{z_k^* B z_k} \quad \frac{z_k^* z_k}{z_k^* B^{-1} z_k}$$

$$\leqq 1 - \frac{4}{((\lambda_M(B)/\lambda_m(B))^{1/2} + (\lambda_m(B)/\lambda_M(B))^{1/2})^2} \cdot$$

This last inequality is due to L. V. Kantorovich [3], and is proved by maximizing the function

$$\left(\sum_{k=1}^{n} \lambda_k u_k^2\right)\left(\sum_{k=1}^{n} \lambda_k^{-1} u_k^2\right),$$

subject to the restraint

$$\sum_{k=1}^{n} u_k^2 = 1.$$

The results here will be stated for A p.d.h. Let $\psi(x)$ be a differentiable function for $x \in I_A$ and we may define $\psi(A)$ as in [1]. For, since A is p.d.h. there exists U, unitary, such that $U^{-1}AU$ is the direct sum of scalar matrices $\lambda_1 I_1, \cdots, \lambda_e I_e$. We then define $U^{-1}\psi(A)U$ as the direct sum of the matrices $\psi(\lambda_1)I_1, \cdots, \psi(\lambda_e)I_e$. We shall assume $\psi(x)$ is real, $\psi^*(A) = \psi(A^*)$, and $\psi(A)$ is nonsingular, $\psi(\lambda(A)) \neq 0$.

Set

$$R_\psi = \psi^*(A)\psi(A) = \psi(A^*)\psi(A)$$

and

$$B_\psi = A^* R_\psi A = A^*\psi(A^*)A\psi(A).$$

R_ψ defines a metric as in (2) since $\psi(\lambda(A)) \neq 0$. Define

(7) $$\mu(B) = \lambda_m(B)/\lambda_M(B),$$

(8) $$\sigma(B) = \frac{4}{((\lambda_M(B)/\lambda_m(B))^{1/2} + (\lambda_m(B)/\lambda_M(B))^{1/2})^2},$$

and

(9) $$\alpha_\psi = \frac{\sigma(B_\psi)}{\sigma(B_1)} \cdot$$

In view of (6) we shall say that R_ψ is as fast a metric as I when $\alpha_\psi \geqq 1$.

THEOREM 1. *If*

$$-\frac{\psi^2(x)}{x} \leqq \psi(x)\psi'(x) \leqq 0 \tag{10}$$

for $x \in I_A$, *then* R_ψ *is as fast a metric as* I.

PROOF. From (7) and (8) we have

$$\sigma(B_\psi) = \frac{4\mu(B_\psi)}{(1+\mu(B_\psi))^2}.$$

Set $f_\psi(x) = x^2\psi^2(x)$ and then

$$B_\psi = A^2\psi^2(A) = f_\psi(A).$$

A p.d.h. and (10) together imply

$$f_\psi'(x) = 2x(\psi^2(x) + x\psi(x)\psi'(x)) \geqq 0$$

and thus $f_\psi(x)$ is monotonic increasing in I_A. Similarly $\psi^2(x)$ is monotonic decreasing in I_A. To simplify the notation set

$$\sigma(B_\psi) = \sigma_\psi, \qquad \omega_\psi = \frac{\psi^2(\lambda_m(A))}{\psi^2(\lambda_M(A))}, \qquad \bar{\mu} = \mu^2(A)$$

and then

$$\begin{aligned}\frac{\sigma_\psi}{4} &= \frac{\lambda_m(f_\psi(A))/\lambda_M(f_\psi(A))}{(1+\lambda_m(f_\psi(A))/\lambda_M(f_\psi(A)))^2} \\ &= \frac{\min_{\lambda(A)} f_\psi(\lambda(A))/\max_{\lambda(A)} f_\psi(\lambda(A))}{\left(1+\min_{\lambda(A)} f_\psi(\lambda(A))/\max_{\lambda(A)} f_\psi(\lambda(A))\right)^2} \\ &= \frac{\bar{\mu}\omega_\psi}{(1+\bar{\mu}\omega_\psi)^2}.\end{aligned}$$

The condition

$$\alpha_\psi \geqq 1 \tag{11}$$

is then equivalent to

$$\frac{\bar{\mu}\omega_\psi}{(1+\bar{\mu}\omega_\psi)^2} \geqq \frac{\bar{\mu}}{(1+\bar{\mu})^2},$$

$$\bar{\mu}^2 \omega_\psi^2 - \omega_\psi(\bar{\mu}^2 + 1) + 1 \leqq 0.$$

Noting that $\bar{\mu} \leqq 1$, (11) becomes

$$1 \leqq \omega_\psi \leqq 1/\bar{\mu}^2$$

which in turn reduces to

$$\mu^4(A)\psi^2(\lambda_M(A)) \leqq \psi^2(\lambda_m(A))\mu^4(A) \leqq \psi^2(\lambda_M(A)). \tag{12}$$

Substituting in the last inequality of (12) yields

$$\psi^2(\mu(A)\lambda_M(A))\mu^4(A) \leqq \psi^2(\lambda_M(A)),$$

$$\frac{\mu^2(A)}{\lambda_M(A)}(\mu(A)\lambda_M(A))^2\psi^2(\mu(A)\lambda_M(A)) \leqq \psi^2(\lambda_M(A)),$$

and finally

$$\mu^2(A)f_\psi(\mu(A)\lambda_M(A)) \leqq f_\psi(\lambda_M(A)). \tag{13}$$

The monotonicity of f_ψ implies (13) and the left side of (12) follows similarly.

For example, if

$$\psi(x) = x^\alpha, \qquad -1 \leqq \alpha \leqq 0,$$

(10) clearly holds.

It is of interest to obtain an estimate of α_ψ solely in terms of ψ and the entries of A.

THEOREM 2. *Assume the conditions of Theorem* 1. *Let* $i_1 \leqq i_2 \leqq \cdots \leqq i_k$, $k \leqq n$, *be a sequence of positive integers and set*

$$S(i_1 \cdots, i_k) = \frac{1}{k}\sum_{\alpha,\nu=1}^{k} A_{i_\alpha i_\nu},$$

$$M = \max S(i_1, \cdots, i_k),$$

$$m = \min S(i_1, \cdots, i_k),$$

$$\mu_0 = (m/M)^2,$$

and

$$\delta = \psi^2(m)/\psi^2(M).$$

Then

$$\alpha_\psi \geqq \frac{\delta}{(1 + \mu_0\delta)^2}.$$

PROOF. Note that

$$\lambda_M(A) = \max_{z^*z=1} z^*Az,$$

$$\lambda_m(A) = \min_{z^*z=1} z^*Az,$$

and hence $z^*Az \in I_A$ for $z^*z=1$. Let z_0 be the unit vector with $1/k^{1/2}$ in the i_j position, and 0 elsewhere. It is easy to check that

$$z_0^*Az_0 = S(i_1, \cdots, i_k) \in I_A.$$

It follows from the monotonicity properties of ψ and f_ψ that

$$\begin{aligned}\alpha_\psi &= \frac{\omega_\psi}{(1/(1+\bar{\mu}) + (\bar{\mu}/(1+\bar{\mu}))\,\omega_\psi)^2} \\ &\geqq \frac{\omega_\psi}{(1+\bar{\mu}\omega_\psi)^2} \\ &\geqq \frac{\psi^2(m)/\psi^2(M)}{(1+f_\psi(m)/f_\psi(M))^2} \\ &= \frac{\delta}{(1+\mu_0\delta)^2}\cdot\end{aligned}$$

We may remark that for

$$\frac{\delta}{(1+\mu_0\delta)^2} \geqq 1$$

it is necessary that $\mu_0 \leqq 1/4$.

References

1. W. L. Ferrar, *Finite matrices*, Oxford, 1951, p. 99.
2. A. S. Householder, *Principles of numerical analysis*, McGraw-Hill, 1953.
3. L. V. Kantorovich, *Functional analysis and applied mathematics*, Uspehi Matematičeskih Nauk vol. 3 no. 6 (1948) pp. 89–185.

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