A NOTE ON TORSION-FREE NIL GROUPS

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Introduction. A *nil group* is an (additively written) abelian group G such that every associative ring R with G as its additive group has trivial multiplication, i.e., $xy=0$ for every $x, y \in R$. Szele [2] has shown that every nil group is either a torsion group (every element of finite order) or a torsion-free group (every element except the identity of infinite order), and he has completely characterized the torsion nil groups. In [3], Szele discusses some questions related to nil groups, but is unable to find conditions for the existence of torsion-free nil groups.

An additive group G is said to be *strongly nil* if there is no nontrivial ring, associative or nonassociative, which has G as its additive group. It is the purpose of this note to characterize those nil and strongly nil groups which are (isomorphic to) a weak direct sum of subgroups of the additive group of rationals, and to show that every such nil group is also strongly nil.

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The basic lemma. We use, in modified form, the characterization of the subgroups of the additive rationals R^+ given by Beaumont and Zuckerman [1]: designate by $p_1, p_2, \cdots, p_j, \cdots$ the primes in their natural order; then for any subgroup K of R^+ which contains the rational integers, let k_j be the greatest non-negative integer n such that p_j^{-n} appears in K, if such an integer exists, and the symbol ∞ otherwise. K is, then, the set of all rationals u/v with u an arbitrary integer and v an arbitrary integer of the form $\prod_j p_j^{n_j}$ with $n_j \leqq k_j$. We denote K by the sequence $(k_1, k_2, \cdots, k_j, \cdots)$; and since every nonzero subgroup of R^+ is isomorphic to one containing the rational integers, this characterization is essentially complete. The initials i.c. will designate "integer containing" with reference to subgroups of R^+.

If $K=(k_1, k_2, \cdots)$, $L=(l_1, l_2, \cdots)$, $M=(m_1, m_2, \cdots)$ are i.c. subgroups of R^+, define (K, L, M) to be the set of all rationals t such that $tKL \subset M$. It is clear that (K, L, M) is a subgroup of M, and we shall

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need conditions making $(K, L, M)=0$. Assume $(K, L, M)\neq 0$, then

(1) if $k_j=\infty$, then $m_j=\infty$,

(2) if $l_j=\infty$, then $m_j=\infty$,

(3) if k_j and l_j are both finite, then $m_j \geqq k_j+l_j$ for almost all j.

For if $k_j=\infty$ and $m_j<\infty$, every element of (K, L, M) would be divisible by p_j^n for arbitrarily high powers n, and this is a contradiction unless $(K, L, M)=0$. The proof of (2) is analogous. If $k_j<\infty$ and $l_j<\infty$, and if there were infinitely many subscripts j for which $m_j<k_j+l_j$, every element of (K, L, M) would have to be divisible by infinitely many primes, yielding again $(K, L, M)=0$. This proves (3).

On the other hand, if (1), (2), (3) hold, and if $m_j \geqq k_j+l_j$ for all j, then (K, L, M) contains 1. Otherwise, if $j_1, \cdots, j_q$ denote those subscripts for which $k_{j_i}<\infty$, $l_{j_i}<\infty$, $m_{j_i}<k_{j_i}+l_{j_i}$, $i=1, \cdots, q$, then

$$\prod_{i=1}^{q} p_{j_i}^{n_i} \in (K, L, M)$$

where $n_i=k_{j_i}+l_{j_i}-m_{j_i}$, and again $(K, L, M)\neq 0$.

LEMMA. $(K, L, M)=0$ *if, and only if,* (1) *or* (2) *or* (3) *is false.*

(As an easy application of this lemma, suppose K is an i.c. nil subgroup of R^+. Then $(K, K, K)=0$, and since (1) and (2) are satisfied, (3) must be false, i.e., $k_j<2k_j$ for an infinite number of subscripts j. Hence, the i.c. nil subgroups of R^+ are given by those sequences $(k_1, k_2, \cdots)$ for which $0<k_j<\infty$ for an infinite number of k_j's. This is essentially Theorem 3 of [1].)

Direct sum case. Let Λ be any index set, and let $\{H_\lambda\}$, $\lambda\in\Lambda$, be a set of i.c. subgroups of R^+. The weak direct sum G of these groups H_λ is defined in terms of a basis $\{a_\lambda\}$, $a_\lambda\in H_\lambda$, as the set of all elements

$$x = \sum_{\lambda} r_\lambda(x) a_\lambda$$

where: $r_\lambda(x)\in H_\lambda$; for each x, $r_\lambda(x)=0$ for almost all $\lambda\in\Lambda$; equality and addition are defined componentwise; $0a_\lambda=0$ and $1a_\lambda=a_\lambda$ for all $\lambda\in\Lambda$. If R is a ring with G as its additive group, then

$$a_\lambda a_\mu = \sum_{\nu} s_{\lambda\mu}^{\nu} a_\nu \in R$$

for suitable structure constants $s_{\lambda\mu}^{\nu}\in H_\nu$, and

$$r_\nu(xy) = \sum_{\lambda,\mu} r_\lambda(x) r_\mu(y) s_{\lambda\mu}^{\nu}.$$

THEOREM 1. *G is strongly nil if and only if* $(H_\lambda, H_\mu, H_\nu)=0$ *for every* λ, μ, ν *in* Λ.

THEOREM 2. *G is nil if and only if* $(H_\lambda, H_\mu, H_\nu)=0$ *for every* λ, μ, ν *in* Λ *satisfying either* (i) $\lambda=\mu=\nu$ *or* (ii) $\lambda\neq\nu$, $\mu\neq\nu$.

PROOFS. Assume $t\neq 0$ is an element of $(H_\lambda, H_\mu, H_\nu)$ for some λ, μ, ν of Λ. For each pair of elements $x, y\in G$, define multiplication by

$$xy = tr_\lambda(x)r_\mu(y)a_\nu$$

and this makes G into a nontrivial ring R (since $a_\lambda a_\mu\neq 0$). Associativity of this multiplication is implied by (i) or (ii):

$$(xy)z = tr_\lambda(tr_\lambda(x)r_\mu(y)a_\nu)r_\mu(z)a_\nu,$$
$$x(yz) = tr_\lambda(x)r_\mu(tr_\lambda(y)r_\mu(z)a_\nu)a_\nu.$$

If $\lambda=\mu=\nu$, each of these becomes $t^2r_\lambda(x)r_\lambda(y)r_\lambda(z)a_\lambda$. If $\lambda\neq\nu$, $\mu\neq\nu$, $r_\lambda(tr_\lambda(x)r_\mu(y)a_\nu)=r_\mu(tr_\lambda(y)r_\mu(z)a_\nu)=0$.

Conversely, let R be a ring over G, and assume $(H_\lambda, H_\mu, H_\nu)=0$ for some λ, μ, ν. It is clear that $s^\nu_{\lambda\mu}=0$, then, from which it follows that G is strongly nil. However, if $s^\nu_{\lambda\mu}=0$ under conditions (i) or (ii), then $a_\lambda a_\lambda=0$ for every λ, and $a_\lambda a_\mu=s^\lambda_{\lambda\mu}a_\lambda+s^\mu_{\lambda\mu}a_\mu$. The associative law imposed on the product $(a_\lambda a_\mu)a_\mu$ yields $s^\mu_{\lambda\mu}=0$, and a similar argument on the product $a_\lambda(a_\lambda a_\mu)$ yields $s^\mu_{\lambda\mu}=0$. Therefore, G is nil.

Remark. It is interesting that for groups which are the weak direct sum of i.c. subgroups of R^+, "nil" implies "strongly nil." For suppose t is a nonzero element of (K, L, K). Then $tKL\subset K$, and so $tKL^2\subset KL$, whence $t^2KL^2\subset tKL\subset K$. It follows that t^2 is an element of (L, L, K). Similarly, $(K, L, L)\neq 0$ implies $(K, K, L)\neq 0$. Thus, the conditions of Theorem 2 imply those of Theorem 1.

REFERENCES

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