A NOTE ON TORSION-FREE NIL GROUPS

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Introduction. A nil group is an (additively written) abelian group $G$ such that every associative ring $R$ with $G$ as its additive group has trivial multiplication, i.e., $xy = 0$ for every $x, y \in R$. Szele [2] has shown that every nil group is either a torsion group (every element of finite order) or a torsion-free group (every element except the identity of infinite order), and he has completely characterized the torsion nil groups. In [3], Szele discusses some questions related to nil groups, but is unable to find conditions for the existence of torsion-free nil groups.

An additive group $G$ is said to be strongly nil if there is no nontrivial ring, associative or nonassociative, which has $G$ as its additive group. It is the purpose of this note to characterize those nil and strongly nil groups which are (isomorphic to) a weak direct sum of subgroups of the additive group of rationals, and to show that every such nil group is also strongly nil.

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The basic lemma. We use, in modified form, the characterization of the subgroups of the additive rationals $\mathbb{R}^+$ given by Beaumont and Zuckerman [1]: designate by $p_1, p_2, \ldots, p_j, \ldots$ the primes in their natural order; then for any subgroup $K$ of $\mathbb{R}^+$ which contains the rational integers, let $k_j$ be the greatest non-negative integer $n$ such that $p_j^n$ appears in $K$, if such an integer exists, and the symbol $\infty$ otherwise. $K$ is, then, the set of all rationals $u/v$ with $u$ an arbitrary integer and $v$ an arbitrary integer of the form $\prod_j p_j^{n_j}$ with $n_j \leq k_j$. We denote $K$ by the sequence $(k_1, k_2, \ldots, k_j, \ldots)$; and since every nonzero subgroup of $\mathbb{R}^+$ is isomorphic to one containing the rational integers, this characterization is essentially complete. The initials i.e. will designate “integer containing” with reference to subgroups of $\mathbb{R}^+$.

If $K = (k_1, k_2, \ldots), L = (l_1, l_2, \ldots), M = (m_1, m_2, \ldots)$ are i.e. subgroups of $\mathbb{R}^+$, define $(K, L, M)$ to be the set of all rationals $t$ such that $tKL \subseteq M$. It is clear that $(K, L, M)$ is a subgroup of $M$, and we shall
need conditions making \((K, L, M)\) = 0. Assume \((K, L, M)\) \(\neq 0\), then

1. if \(k_j = \infty\), then \(m_j = \infty\),
2. if \(l_j = \infty\), then \(m_j = \infty\),
3. if \(k_j\) and \(l_j\) are both finite, then \(m_j \geq k_j + l_j\) for almost all \(j\).

For if \(k_j = \infty\) and \(m_j < \infty\), every element of \((K, L, M)\) would be divisible by \(p_j^n\) for arbitrarily high powers \(n\), and this is a contradiction unless \((K, L, M) = 0\). The proof of (2) is analogous. If \(k_j < \infty\) and \(l_j < \infty\), and if there were infinitely many subscripts \(j\) for which \(m_j < k_j + l_j\), every element of \((K, L, M)\) would have to be divisible by infinitely many primes, yielding again \((K, L, M) = 0\). This proves (3).

On the other hand, if (1), (2), (3) hold, and if \(m_j \geq k_j + l_j\) for all \(j\), then \((K, L, M)\) contains 1. Otherwise, if \(j_1, \ldots, j_q\) denote those subscripts for which \(k_{j_i} < \infty\), \(l_{j_i} < \infty\), \(m_{j_i} < k_{j_i} + l_{j_i}\), \(i = 1, \ldots, q\), then

\[
\prod_{i=1}^{q} p_{j_i}^{n_{j_i}} \in (K, L, M)
\]

where \(n_i = k_{j_i} + l_{j_i} - m_{j_i}\), and again \((K, L, M)\) \(\neq 0\).

**Lemma.** \((K, L, M) = 0\) if, and only if, (1) or (2) or (3) is false.

(As an easy application of this lemma, suppose \(K\) is an i.e. nil subgroup of \(R^+\). Then \((K, K, K) = 0\), and since (1) and (2) are satisfied, (3) must be false, i.e., \(k_j < 2k_j\) for an infinite number of subscripts \(j\). Hence, the i.e. nil subgroups of \(R^+\) are given by those sequences \((k_1, k_2, \ldots)\) for which \(0 < k_j < \infty\) for an infinite number of \(k_j\)'s. This is essentially Theorem 3 of [1].)

**Direct sum case.** Let \(\Lambda\) be any index set, and let \(\{H_\lambda\}, \lambda \in \Delta\), be a set of i.c. subgroups of \(R^+\). The weak direct sum \(G\) of these groups \(H_\lambda\) is defined in terms of a basis \(\{a_\lambda\}, a_\lambda \in H_\lambda\), as the set of all elements

\[
x = \sum_\lambda r_\lambda(x) a_\lambda
\]

where: \(r_\lambda(x) \in H_\lambda\); for each \(x\), \(r_\lambda(x) = 0\) for almost all \(\lambda \in \Delta\); equality and addition are defined componentwise; \(0a_\lambda = 0\) and \(1a_\lambda = a_\lambda\) for all \(\lambda \in \Delta\). If \(R\) is a ring with \(G\) as its additive group, then

\[
a_\lambda a_\mu = \sum_\nu s_{\lambda\mu}^\nu a_\nu \in R
\]

for suitable structure constants \(s_{\lambda\mu}^\nu \in H_\nu\), and

\[
r_\nu(xy) = \sum_{\lambda, \mu} r_\nu(x) r_\mu(y) s_{\lambda\mu}^\nu.
\]
Theorem 1. $G$ is strongly nil if and only if $(H_{\lambda}, H_{\mu}, H_{\nu}) = 0$ for every $\lambda, \mu, \nu$ in $\Lambda$.

Theorem 2. $G$ is nil if and only if $(H_{\lambda}, H_{\mu}, H_{\nu}) = 0$ for every $\lambda, \mu, \nu$ in $\Lambda$ satisfying either (i) $\lambda = \mu = \nu$ or (ii) $\lambda \neq \nu, \mu \neq \nu$.

Proofs. Assume $t \neq 0$ is an element of $(H_{\lambda}, H_{\mu}, H_{\nu})$ for some $\lambda, \mu, \nu$ of $\Lambda$. For each pair of elements $x, y \in G$, define multiplication by $xy = tr_{\lambda}(x)r_{\mu}(y)a_{\nu}$ and this makes $G$ into a nontrivial ring $R$ (since $a_{\lambda}a_{\mu} \neq 0$). Associativity of this multiplication is implied by (i) or (ii):

$$(xy)z = tr_{\lambda}(tr_{\mu}(x)r_{\mu}(y)a_{\nu})r_{\lambda}(x)a_{\mu},$$

$$x(yz) = tr_{\lambda}(x)r_{\mu}(tr_{\lambda}(y)r_{\mu}(z)a_{\nu})a_{\mu}.$$ 

If $\lambda = \mu = \nu$, each of these becomes $t^{2}r_{\lambda}(x)r_{\lambda}(y)r_{\lambda}(z)a_{\lambda}$. If $\lambda \neq \nu, \mu \neq \nu$, $r_{\lambda}(tr_{\lambda}(x)r_{\lambda}(y)a_{\nu}) = r_{\lambda}(tr_{\lambda}(y)r_{\mu}(z)a_{\nu}) = 0$.

Conversely, let $R$ be a ring over $G$, and assume $(H_{\lambda}, H_{\mu}, H_{\nu}) = 0$ for some $\lambda, \mu, \nu$. It is clear that $s_{\lambda\mu} = 0$, then, from which it follows that $G$ is strongly nil. However, if $s_{\lambda\mu} = 0$ under conditions (i) or (ii), then $a_{\lambda}a_{\mu} = 0$ for every $\lambda$, and $a_{\lambda}a_{\mu} = s_{\lambda\mu}a_{\lambda} + s_{\lambda\mu}a_{\mu}$. The associative law imposed on the product $(a_{\lambda}a_{\mu})a_{\nu}$ yields $s_{\lambda\mu} = 0$, and a similar argument on the product $a_{\lambda}(a_{\lambda}a_{\mu})$ yields $s_{\lambda\mu} = 0$. Therefore, $G$ is nil.

Remark. It is interesting that for groups which are the weak direct sum of i.c. subgroups of $R^{+}$, "nil" implies "strongly nil." For suppose $t$ is a nonzero element of $(K, L, K)$. Then $tKL \subseteq K$, and so $t^{2}KL \subseteq K$, whence $t^{2}KL \subseteq tK \subseteq K$. It follows that $t^{2}$ is an element of $(L, L, K)$. Similarly, $(K, L, L) \neq 0$ implies $(K, K, L) \neq 0$. Thus, the conditions of Theorem 2 imply those of Theorem 1.

References


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