ON A BLOCH-LANDAU CONSTANT

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As usual, denote by S the class of functions, $w=f(z)=z+a_2z^2+\cdots$, regular and schlicht for $|z|<1$. Let T be the subset of S for which

$$(1-|z|^2)|f'(z)| \leqq 1, \qquad |z|<1. \tag{1}$$

Let D_f denote the image of $\{|z|<1\}$ under $f(z)$.

THEOREM. *$f\in T \rightarrow D_f$ contains the circle* $\{|w|<.569\}$.

PROOF. Inequality (1) implies (cf. [1])

$$a_2 = 0, \qquad |a_3| \leqq 1/3, \tag{2}$$

and

$$|f(z)| \leqq \frac{1}{2}\log\frac{1+|z|}{1-|z|} = |z|M(|z|). \tag{3}$$

For fixed t, $0<t<1$, put $f(z,t)=f(tz)/t$. Thus

$$|f(z,t)| \leqq M(t), \qquad |z|<1.$$

Now let

$$\tilde{f}(z,t) = M(t)\left[\phi\left\{\left(\frac{f(z,t)}{M(t)}\right)^3\right\}\right]^{1/3} = z + a_3t^2z^3+\cdots,$$

where $\phi(z)=z/(1+z)^2$. If $f(z)$ omits $\gamma>0$, then $\tilde{f}(z,t)$ omits

$$\gamma(t) = (\gamma/t)[1+\gamma^3/t^3M^3(t)]^{-2/3}. \tag{4}$$

Let

$$g(z,t) = \frac{\tilde{f}(z,t)}{(1-\tilde{f}(z,t)/\gamma(t))} = z + b_2z^2 + b_3z^3+\cdots, \qquad |z|<1.$$

We have

$$b_2 = 1/\gamma(t), \qquad b_3 = 1/\gamma^2(t) + a_3t^2, \qquad |a_3| \leqq 1/3.$$

Now, since $g(z,t)$ is regular and schlicht for $|z|<1$, it follows [2; 3] that

$$|b_3-\alpha b_2^2| \leqq 2e^{-2\alpha/(1-\alpha)}+1 \quad \text{for all } \alpha\in(0,1).$$

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This means

$$(5) \qquad |\gamma(t)| \geqq \left[\frac{3(1-\alpha)}{6 \exp\{-2\alpha/(1-\alpha)\} + 3 + t^2}\right]^{1/2}, \quad 0 < \alpha < 1.$$

In particular, when $\alpha = .372947$, $t = .99925$, a numerical computation shows that (4) and (5) jointly imply $|\gamma| > .569$.

COROLLARY. *$f \in S \rightarrow D_f$ contains a circle of radius $> .569$, i.e. the Bloch-Landau constant $\mathfrak{A}$ [1] satisfies the inequality $\mathfrak{A} > .569$.*

PROOF. As pointed out by Landau [1], it is sufficient to consider functions of class T for the purpose of obtaining a lower bound on $\mathfrak{A}$.

The bound, .569, obtained here is only very slightly better than Landau's original bound of .566. The point of main interest is that it becomes clear that in order to improve Landau's bound one must make use of (1) globally for $|z| < 1$, instead of just in the neighborhood of $z = 0$, as Landau did.

REFERENCES

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