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A NOTE ON RECURRENT FLOWS

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Let R be a metric space and let $f\colon R \times \{t \,|\, t \text{ is real}\} \to R$ be a continuous flow in R. The flow f is said to be *recurrent* (in the sense of Gottschalk and Hedlund) provided that if ϵ and s are positive real numbers, then there exists a real number t such that $t > s$ and $\rho(x, f(x, t)) < \epsilon$ for all $x \in R$.

THEOREM. *Consider the statements*:

(I) f *is recurrent*,

(II) *There exists a sequence* $t_1 < t_2 < \cdots \to +\infty$ *of real numbers such that* $\lim_{n \to \infty} f(A, t_n) = A$ *for every closed subset* A *of* R,

(III) *There exists a sequence* $t_1 < t_2 < \cdots \to +\infty$ *of real numbers such that* $\limsup_{n \to \infty} f(A, t_n) \subset A$ *for every closed subset* A *of* R.

Then,

(1) I *implies* II; II *implies* III,

(2) *If* R *is compact, then* I, II, III *are equivalent.*

PROOF. Assume I. There exists a sequence $t_1 < t_2 < \cdots \to +\infty$ of real numbers such that $\rho(x, f(x, t_n)) < 1/n$ for all $x \in R$ and all positive integers n. Let A be a closed subset of R. If $x \in A$, then $f(x, t_n) \to x$. Hence $A \subset \liminf_{n \to \infty} f(A, t_n)$. Suppose $x \in \limsup_{n \to \infty} f(A, t_n)$. There exist $x_1, x_2, \cdots \in A$ and a sequence $n_1 < n_2 < \cdots \to +\infty$ of positive integers such that $\rho(x, f(x_i, t_{n_i})) \to 0$. Since $\rho(x_i, f(x_i, t_{n_i})) < 1/n_i$ for all positive integers i, it follows that $\rho(x_i, f(x_i, t_{n_i})) \to 0$ whence $x_i \to x$ and $x \in A$. Hence $\limsup_{n \to \infty} f(A, t_n) \subset A$. This proves that I implies II.

It is immediate that II implies III.

Assume III. Let R be compact. Let ϵ and s be positive real numbers. There exists a finite sequence $A_1, \cdots, A_k$ of closed subsets of R such that $A_1 \cup \cdots \cup A_k = R$ and diam $(A_i) < \epsilon/2$ $(i=1, \cdots, k)$. Choose a positive integer n so that $t_n > s$ and $f(A_i, t_n) \subset N(A_i, \epsilon/2)$ $(i=1, \cdots, k)$ where $N(A, \delta)$ denotes the δ-neighborhood of the set A. Let $x \in R$. Then $x \in A_j$ for some j $(j=1, \cdots, k)$. It follows that $\rho(f(x, t_n), y) < \epsilon/2$ for some $y \in A_j$. Since $x, y \in A_j$ we have $\rho(x, y) < \epsilon/2$. Hence, $\rho(x, f(x, t_n)) < \epsilon$. This proves (2).

We remark that the above theorem and proof apply to discrete flows. Also the theorem answers a question by Nemyckii concerning a characterization of pseudo-minimal sets (cf. V. V. Nemyckii, *Topological problems of the theory of dynamical systems*, American Mathematical Society Translation, no. 103, p. 69).

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