Introduction. This note contains a method for constructing chain-homotopy operators suitable for the de Rham cohomology theory. In particular, it is proved that differentiably homotopic maps induce chain homotopic chain-mappings in the exterior algebra of differential forms (Formula 13 below; cf. pp. 80–81 of [1], where the same formula is obtained). This shows that the de Rham theory satisfies the "homotopy axiom" in the sense of S. Eilenberg and N. E. Steenrod (cf. [2]); hence the de Rham cohomology groups of a differentiably contractible manifold are trivial. This fundamental result is often referred to as the "Poincaré Lemma."

A simple generalization is given in the case of an almost product structure.

Almost complex and complex structures are investigated in §5; no genuine chain-homotopies are obtained, and in §6 is given an example which shows that $\bar{\partial}$-cohomology does not satisfy the homotopy axiom, even in the case of complex manifolds and analytic homotopies; this example is due to Professor K. Kodaira.

1. Definitions and notations. By "manifold" we mean "differentiable manifold of class $C^\infty,"$ by "map," "map of class $C^\infty,"$ etc.; and all notions such as tangent vector or differential form will be taken in their $C^\infty$-sense. Tangent vectors will always be taken to have been defined by the $C^\infty$-analogue of the definition given in §IV, Chap. II of [10].

If $U$ is a manifold, we denote by $T^1(U)$ the tangent bundle, by $T(U) = \bigoplus_{p=0}^\infty T^p(U)$ the bundle of exterior algebras of tangent vectors. Note that $T^0(U) = \mathbb{R}$ = the reals. By $\Phi(U) = \bigoplus_{p=0}^\infty \Phi^p(U)$ we denote the exterior algebra of differential forms; for our purposes, the most convenient definition is

$$\Phi^p(U) = \text{Hom}_R(U) [\times T^p(U), R(U)]$$

where $R(U) = \Phi(\mathbb{R}) = R$-module of $C^\infty$-maps $U \to R$, and $\times T^p(U)$ denotes the $R(U)$-module of cross-sections of $T^p(U)$. \{If $\Lambda$ is a commutative ring and $A, B$ are $\Lambda$-modules, $\text{Hom}_\Lambda (A, B)$ denotes the $\Lambda$-module of $\Lambda$-homomorphisms $A \to B$.\}

If $v, v' \in \times T^p(U)$ are such that $v| V = v'| V$ ("on $V\"$), where $V$
is some open set of $U$, it is easy to see that $\phi v = \phi v'$ on $V$ for $\phi \in \Phi^p(U)$. Hence the definition of $\Phi^p(U)$ is a "local" one; and $\phi \in \Phi^p(U)$ can be given by giving its values on germs of cross-section; a germ of cross-section at $x \in U$ is the equivalence class of all cross-sections which agree (pairwise) in some neighborhood of $x$.

If $\phi \in \Phi^{p+1}(U)$ and $v \in \mathcal{X}^p(U)$ we define the contraction $v \cdot \phi \in \Phi^q(U)$ by

$$(v \cdot \phi)v' = \phi(v \wedge v')$$

where $v' \in \mathcal{X}^q(U)$.

The exterior derivative $d : \Phi^p \to \Phi^{p+1}$ is given by the formula

$$(d\phi)(v_1 \wedge \cdots \wedge v_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i+1} v_i (\phi(v_1 \wedge \cdots \hat{v}_i \cdots \wedge v_{p+1}))$$

$$+ \sum_{i < j} (-1)^{i+j+1} \phi([v_i, v_j] \wedge v_1 \wedge \cdots \hat{v}_i \cdots \hat{v}_j \cdots \wedge v_{p+1})$$

where the $v_i$ are germs of $\mathcal{X}^1(U), \phi \in \Phi^p(U)$, $[v_i, v_j] = v_iv_j - v_jv_i$ and $\cdots \hat{v} \cdots$ denotes the omission of the term with index $i$. The following will be useful:

**Lemma 1.** The homomorphism $d$ is uniquely characterized by:

(i) If $\phi \in \Phi^p(U), v \in \mathcal{X}^1(U)$, $(d\phi)v = \phi v$,

(ii) If $\phi \in \Phi^p(U)$, $d^2\phi = 0$,

(iii) If $\phi \in \Phi^p(U), \psi \in \Phi(U), d(\phi \wedge \psi) = d\phi \wedge \psi + (\psi)\phi \wedge d\psi$.

Since locally $\Phi^1(U)$ is (isomorphic to) the Grassmann algebra generated by $\Phi^1(U)$ regarded as an $\mathbb{R}(U)$-module, (ii) and (iii) imply (ii'):

$d^2 = 0$.

If $U, V$ are manifolds, and $f : U \to V$ is a map, we denote by $f* : T(U) \to T(V)$ and $f** : \Phi(V) \to \Phi(U)$ the corresponding induced maps.

If $c$ is a differentiable (i.e., $C^\infty$) $p$-chain in $U$ and $\phi \in \Phi^p(U)$, we shall write $\phi \cdot c = \int \phi$. Stokes's theorem then takes the form $(df) \cdot c = \phi \cdot b$, where $b$ denotes the boundary operator of the singular theory.

2. **Almost product structure.** We say that the manifold $U$ has almost product structure $(P, Q)$ if there are homomorphisms $P, Q : T^1(U) \to T^1(U)$ such that $T^1(U) = PT^1(U) \oplus QT^1(U)$ (direct sum). Thus for $v \in T^1(U), v = Pv + Qv$ and hence if $v_i \in T^1(U)$ ($i = 1, \cdots, p$), then $v_1 \wedge \cdots \wedge v_p$ is a sum of terms each of which is the exterior

More accurately: $\Phi(V)$ is the Grassmann algebra generated by $\Phi^p(V)$ for sufficiently small neighbourhoods $V \subset U$.

For fibre bundles the fibres of which are modules, a homomorphism is a fibre-preserving map which, restricted to any fibre, is a homomorphism in the algebraic sense.
product of \( r \) vectors of type \( P_\nu \), by \( s \) vectors of type \( Q_\nu \) where \( r + s = p \); for given \( r, s \) such a term is called a vector of “type (\( r, s \))”; and this process defines a unique projection operator

\[
\prod_{r,s} : T^{r+s}(U) \rightarrow T^{r+s}(U)
\]

onto the submodule of vectors of type \( (r, s) \).

We then define projection operators \( \prod_{r,s} : \Phi^{r+s}(U) \rightarrow \Phi^{r+s}(U) \) by

\[
\prod_{r,s} \phi = \phi \prod_{r,s}.
\]

If \( \phi = \prod_{r,s} \phi \) we say that \( \phi \) is of type \( (r, s) \) (cf. [8]).

Let \( U, V \) be manifolds with almost product structures \((P, Q), (\overline{P}, \overline{Q})\) respectively. A map \( F: U \rightarrow V \) is said to be of type \((l, m)\) (in relation to these structures) if

\[
\prod_{r+l, s+m} F_* = F_* \prod_{r,s}.
\]

A map of type \((0,0)\) is said to be admissible; the same definitions apply to any homomorphisms \( T(U) \rightarrow T(V) \) or \( \Phi(V) \rightarrow \Phi(U) \).

An examination of formula (2) shows that \( d = d'_2 + d'_1 + d''_1 + d''_2 \) where \( d'_2, d'_1, d''_1, d''_2 \) are of types \((2, -1), (1, 0), (0, 1), (-1, 2)\) respectively. \( d_2 = 0 \) leads to the following identities:

\[
\begin{align*}
\left(\text{3}\right) \quad d'_2 &= d'_2 d'_1 + d'_1 d'_2 = d'_2 d''_1 + (d'_2)^2 + d''_1 d'_2 \\
&= d''_1 d'_1 + d''_1 d'_2 + d'_1 d''_2 + d''_2 d'_1 \\
&= d''_1 d''_2 + d'_1 d''_1 = d''_2 d''_2 + (d''_1)^2 + d'_1 d'_2 = (d''_2)^2 = 0.
\end{align*}
\]

In analogy to Lemma 1, we now define the \( R(U) \)-homomorphism \( d_p: \Phi^p(U) \rightarrow \Phi^{p+1}(U) \) by

(i) If \( \phi \in \Phi^p(U) \) and \( v \in \times T^1(U) \), then

\[
(d_p \phi)v = (Pv)\phi.
\]

(ii) If \( \phi \in \Phi^p(U) \), \( (d_p^2 + d d_p)\phi = 0 \).

(iii) If \( \phi \in \Phi^p(U) \) and \( \psi \in \Phi(U) \),

\[
d_p(\phi \wedge \psi) = d_p \phi \wedge \psi + (-1)^r \phi \wedge d_p \psi.
\]

It easily follows that

(ii') \( d_p d + d d_p = 0 \).

It is easily verified that \( 2d''_2 + d'_1 - d''_1 \) satisfies these conditions; whence

\[
\text{(4)} \quad d_p = 2d''_2 + d'_1 - d''_1.
\]

Writing also

\[
\text{(5)} \quad d_q = 2d''_2 + d'_1 - d'_2,
\]
we see that $d = d_P + d_Q$ and, by symmetry, that $d_Q$ is related to $Q$ as $d_P$ is to $P$.

Using (3), (4), we see that $d_P = d_P^2 + 2(d'_1 + d''_1)$ (4, 4'). Hence, noting $d_P^2(U) = d_P + d_P''$ and appealing to Lemma 1, we have

**Lemma 2.** $d_P^2 = 0$ if and only if $d = d'_1 + d''_1$; i.e., $d'_1 = d''_1 = 0$; i.e., $d_P = d'_1$, $d_Q = d''_1$.

It is not hard to prove that the conditions of Lemma 2 are equivalent to the "integrability" of the given almost product structure in which case we have a local product structure.

3. **The $/ \Delta$ operation.** Let $U$, $V$ be manifolds. An obvious almost product structure is defined on $U \times V$ by regarding $P$, $Q$ as the (natural) projection operators associated with the direct sum decomposition $T^1(U \times V) = T^1(U) \oplus T^1(V)$. We shall thus regard vector fields in $U$, $V$ as lying, in an evident manner, in $U \times V$. It is clear that the conditions of Lemma 2 pertain; we write $d_U = d_P$, $d_V = d_Q$. $d_U$ corresponds to "differentiation in $U$ only."

Now, let $\phi \in \Phi^{p+1}(U \times V)$ and let $c$ be a singular $r$-chain in $U$. Then (using a notation due to N. E. Steenrod, cf. [3]) we define $\phi/c \in \Phi^r(V)$ by

$$\phi/c(v)(y) = (-1)^{r}(j_v^*(v \cdot \phi)) \cdot c$$

where $v \in X^r(V)$, $y \in V$ and $j_v: U \rightarrow U \times V$ is the map $x \rightarrow (x, y)$. Then, as is easily seen,

$$(-1)^{r}d(\phi/c) = (-1)^{r+1}d_V\phi/c$$

Also, if $v' \in X^{r+1}(V)$ we have

$$(-1)^{e+1}(d_U\phi/c)(v')(y) = j_v^*(v' \cdot \phi) \cdot c$$

$$= (-1)^{e+1}d(v' \cdot \phi) \cdot c = (-1)^{e+1}j_v^*(v' \cdot \phi) \cdot bc$$

$$= (-1)^{e+1}(\phi/c)(v')(y).$$

Hence

$$d(\phi/c) = \phi(bc) = (-1)^{e}(\phi/c)$$

(cf. 2.9 in [3]).

Now assume that $V$ has almost product structure $(P, Q)$ and that $U = U_P \times U_Q$. Define an almost product structure $(\overline{P}, \overline{Q})$ on $U \times V$ by

$$\overline{P}T^1(U \times V) = T^1(U_P) \oplus PT^1(V),$$

$$\overline{Q}T^1(U \times V) = T^1(U_Q) \oplus QT^1(V).$$
In this situation, formula (8) splits up into various components. We discuss one special case, namely that when $c = c' \times x_0$ where $c'$ is an $r$-chain in $U_P$ and $x_0$ is a point of $U_Q$ regarded as a 0-chain. In this case the homomorphism $\phi \rightarrow \phi/c$ is of type $(-r, 0)$ in relation to the almost product structures $(P, Q), (P, Q)$. By examining (8) in terms of its components we obtain:

\[
\begin{align*}
\delta_i' \phi/c &= (-1)^r \delta_i' (\phi/c), \\
\delta_i' \phi/c &= (-1)^r \delta_i' (\phi/c), \\
\delta_i' \phi/c &= (-1)^r \delta_i' (\phi/c), \\
\delta_i' \phi/c &= (-1)^r \delta_i' (\phi/c),
\end{align*}
\]

from which, using (4) and (5), we obtain

\[
\begin{align*}
d \phi/c - \phi/\psi &= (-1)^r d \phi/c, \\
d \phi/c &= (-1)^r d \phi/c.
\end{align*}
\]

4. Chain homotopies. Let us retain the notations of §3, let $W$ be a third manifold and $F: U \times V \rightarrow W$ a map. We define $\lambda: \Phi(W) \rightarrow \Phi(V)$ by

\[
\lambda \psi = (-1)^{r+1} (F^* \psi)/c
\]

for $\psi \in \Phi(W)$. Then, using (8) we get

\[
(d \lambda + (-1)^{r+1} d \lambda) \psi = (F^* \psi)/bc.
\]

Now, consider the case when $c: I \rightarrow U$ is a singular 1-simplex and define $f_1: V \rightarrow W$ by $f_1(y) = F(c(t), y)$; then $F$ represents a homotopy, and (12) becomes

\[
d \lambda + \lambda d = f_1^* - f_0^*
\]

showing that differentiably homotopic maps induce chain-homotopic homomorphisms.

Next, consider the almost product structures introduced in the second part of §3, and assume that $F$ is admissible (in relation to these structures). The homomorphism $\lambda$ defined by (11) in terms of an $r$-chain $c$ "in $U_P$" will be denoted by $\lambda_P$. Using (10), we get

\[
\begin{align*}
(d \lambda_P + (-1)^{r+1} \lambda_P d_P) \psi &= F^* \psi/bc, \\
\lambda_P^* d_P + (-1)^{r+1} \lambda_P d_Q &= 0
\end{align*}
\]

and finally, in analogy to (13),

\[
\begin{align*}
d \lambda_P + \lambda_P d_P &= f_1^* - f_0^*
\end{align*}
\]

in other words: A homotopy consistent with a given almost product
structure induces chain-homotopies for the operator \( d_P \); and similarly for \( d_Q \).

5. **Almost complex structure** (cf. \([5; 6]\)). Let \( M \) be an \( m \)-manifold, and let \( C^*(M) = T(M) \otimes \mathbb{R} \mathbb{C} \) where \( \mathbb{C} \) are the complex numbers; and let \( C(M) = \mathbb{C} \)-module of \( \mathbb{C} \)-maps \( M \rightarrow \mathbb{C} \). We define

\[
C\Phi^p(M) = \text{Hom}_{C(M)}[\times C^p(M), C(M)]
\]

and \( C\Phi(M) = \sum_{p=0}^{\infty} C\Phi^p(M) \); cf. (1). We also define \( d: C\Phi^p(M) \rightarrow C\Phi^{p+1}(M) \) by the formal analogue of (2); the definitions of \( f_\ast, f^\ast \) are similarly extended. It is clear that the whole “complex” theory is analogous to the “real” theory; also, \( C(M) \) is naturally isomorphic to \( R(M) \otimes_R \mathbb{C} \), \( C\Phi^p(M) \) to \( \Phi^p(M) \otimes_R \mathbb{C} \) and, under this isomorphism, \( d \) corresponds to \( d \otimes 1 \).

We say that \( M \) has a complex almost product structure if there are \( C(M) \)-homomorphisms \( P, Q: C^1(M) \rightarrow C^1(M) \) such that \( C^1(M) = P C^1(M) \oplus Q C^1(M) \), \( P, Q \) being projections. It is clear that the theory of almost product structures (§2 above) has an exact analogue in this situation: and we take over, without change, the definitions of \( \prod_{\ast r}, \prod_{\ast s}, \text{“type (r, s),” d = d_P + d_Q }, \) complete with Lemma 2.

We say that \( M \) has almost complex structure if it has complex almost product structure together with an isomorphism \( k: C^1(M) \rightarrow C^1(M) \) such that \( k P C^1(M) = Q C^1(M), k Q C^1(M) = P C^1(M) \), \( k^2 = 1 \). Then \( k \) can be extended to \( k: C^1(M) \rightarrow C^1(M) \) (and with a slight abuse of notation!) \( k: C\Phi(M) \rightarrow C\Phi(M) \). We write \( kv = \bar{v}, k\phi = \bar{\phi} \). In this case, in accordance with the usual notation, we write \( \partial, \bar{\partial} \) for \( d_P, d_Q \). If \( \bar{\partial}^2 = 0 \) (cf. Lemma 2) the given almost complex structure is called integrable (cf. \([5]\)).

It is well known that if \( M \) has almost complex structure and \( n \) complex dimensions, then it can be assigned a Hermitian metric (cf. \([4, p. 209]\)) and in terms of this a duality operator \( \ast: C\Phi^p(M) \rightarrow C\Phi^{2n-p}(M) \) and a scalar product \( (\phi, \psi) \) for \( \phi, \psi \in C\Phi^p(M) \); cf \([1; 5; 7; 8]\). These operations satisfy

\[
\left( \prod_{r,s}^* \phi, \psi \right) = \left( \phi, \prod_{r,s}^* \psi \right),
\]

\[
\ast \prod_{r,s}^* = \prod_{r,s}^* \ast
\]

and also, writing \( \partial = - \ast \partial \ast \),

\[
(\phi, \partial \psi) = (\bar{\partial} \phi, \psi)
\]

if \( \phi, \psi \) are forms with compact carriers (cf. \([5]\)). We define
\[ (17) \quad \Delta = 2(\partial \bar{\partial} + \bar{\partial} \partial). \]

Now, let \( U \) be a subdomain (i.e., an open set) of \( M \) such that the closure of \( U \) in \( M \) is compact. By \( \mathcal{L} \) denote the Hilbert space (in terms of the scalar product just introduced) of norm-finite differential forms on \( U \) and by \( \mathcal{J} \) the space of forms \( \phi \in C\Phi(M) \) such that \( \partial \phi = \bar{\partial} \phi = 0 \) and \( \phi = 0 \) outside \( U \); then \( \mathcal{J} \) can be regarded as a subspace of \( \mathcal{L} \); we denote by \( F: \mathcal{L} \to \mathcal{J} \) the associated projection operator. There exists a "Green's operator" \( G: \mathcal{L} \to \mathcal{L} \) such that
\[ (18) \quad \Delta G\phi = \phi - F\phi \]
(cf. [5]).

Define \( H, J: \mathcal{L} \to \mathcal{L} \) by
\[ (19) \quad H = 2\partial(\partial G - G\partial) + F, \]
\[ J = 2\bar{\partial}G. \]

If the structure is complex, \( \partial^2 = 0, \bar{\partial}^2 = 0 \) and hence \( \partial \Delta = \bar{\partial} \Delta, \bar{\partial} \Delta = \partial \Delta; \) hence in this case \( \Delta H = 0 \). If \( U \) is a closed, compact manifold, \( \partial G - G\bar{\partial} = 0 \) and \( H = F \).

In the case of complex euclidean \( n \)-space, it is trivial that there exists a Green's operator \( G \) satisfying \( \Delta G\phi = \phi \) and, if \( \phi \) has a compact support, \( \partial G\phi = G\partial\phi \). Hence, if we assume that \( U \) is an arbitrary subdomain of complex euclidean space, then \( H\phi = 0 \) provided that the support of \( \phi \) is compact relative to \( U \).

As is easily verified,
\[ (20) \quad \partial J + J\partial = I - H. \]

Let \( V \) be another almost complex manifold; give to \( U \times V \) the natural induced almost complex structure; and by \( J_U, H_U, \partial_U \) denote the operators on \( C\Phi(U \times V) \) associated with \( U \). Then, if \( \phi \) is some singular \( r \)-chain in \( U \), define \( L: C\Phi(U \times V) \to C\Phi(V) \) by
\[ (21) \quad L\phi = J_U \phi/\partial. \]

It is easily seen that
\[ (22) \quad (-1)^{r+1}\partial L + L\partial = L\partial_U \]
or, using (20),
\[ (23) \quad ((-1)^{r+1}\partial L + L\partial)\phi = (I - H_U - \partial_U J_U)\phi/\partial. \]

Notice that, since there is no Stokes's formula in the geometrical sense for \( \partial \), no formula analogous to (8) can be obtained; similarly, no formulas analogous to (10) seem to exist, as singular chains cannot be closely related to almost complex structure.
In particular, if $r = 0$, $\partial d J_U/c = 0$ and (23) becomes

$$(-\bar{\partial}L + L\bar{\partial})\phi = (I - H_U)\phi/c.$$  

Let $W$ be a third almost complex manifold, $F: U \times V \to W$ a map such that $\partial F^* = F^* \bar{\partial}$, let $c = u_1 - u_0$ where $u_0, u_1 \in U$, write $f_*(v) = F(u_1, v)$, and $\lambda = LF^*: \mathbb{C} \Phi(W) \to \mathbb{C} \Phi(V)$. Then, (24) gives

$$(-\bar{\partial}\lambda + \lambda \partial)\phi = (f_1^* - f_0^*)\phi - (H_U F^*\phi)_1 + (H_U F^*\phi)_0$$

where $(H_U F^*\phi)_i = H_U F^*\phi|_{u_1 \times V}$.

If $U$ is compact and connected, $H_U = F_U$ where $F_U$ is the projection onto the space of forms satisfying $d_U\phi = \bar{\partial}U\phi = 0$, and $\mathcal{J}_U$ (subspace of such forms of degree 0) is isomorphic to $C$; then $(H_U F^*\phi)_1 = (H_U F^*\phi)_0$ and (25) becomes

$$-\bar{\partial}\lambda + \lambda \partial = f_1^* - f_0^*.$$  

6. An example (cf. the Introduction). Let $G$ be the multiplicative group consisting of matrices of the form

$$z = \begin{bmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{bmatrix}$$

where $z_i \in C$; let $D$ be the (discrete) subgroup consisting of all $z$ such that $z_i$ are Gaussian integers. Then $V = G/D$ (i.e., the space of right cosets $z \cdot D$) is a homogeneous compact complex manifold (which was first considered by Iwasawa). It is easily seen that (in classical notation) the holomorphic 1-forms

$$w_1 = dz_1, \quad w_2 = dz_2 - z_3dz_1, \quad w_3 = dz_3$$

are right invariant on $G$; they can hence be regarded as holomorphic 1-forms on $V$. Further, it is not hard to verify that $w_1, w_2, w_3$ generate the $\bar{\partial}$-homology group $H^{1,0}_\delta(V)$ of forms of type $(1, 0)$. Hence

$$\dim H^{1,0}_\delta(V) = 3.$$  

By the duality theorem of Kodaira-Serre (cf. [9; 11])

$$\dim H^{2,3}_\delta(V) = 3.$$  

It is easy to verify that

$$\psi_1 = w_2 \wedge w_3 \wedge w_1 \wedge w_2 \wedge w_3,$$
$$\psi_2 = w_3 \wedge w_1 \wedge w_1 \wedge w_2 \wedge w_3,$$
$$\psi_3 = w_1 \wedge w_2 \wedge w_1 \wedge w_2 \wedge w_3.$$
represent linearly independent elements of $H^{2,3}(V)$ and hence generate this group.

Now, every $t \in G$ induces the analytic homeomorphism $T_t: z \mapsto t \cdot z$ of $V$ onto itself; obviously each $T_t$ is homotopic to the identity. We have

$$(T_t)^*w_1 = w_1,$$
$$(T_t)^*w_2 = w_2 - l_3w_1 + l_1w_3,$$
$$(T_t)^*w_3 = w_3$$

and hence

$$(T_t)^*\psi_1 = \psi_1 + l_3\psi_2,$$
$$(T_t)^*\psi_2 = \psi_2,$$
$$(T_t)^*\psi_3 = \psi_3 - l_1\psi_2$$

showing that there is no chain-homotopy.

REFERENCES


Birkbeck College and
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