CHAIN HOMOTOPY AND THE de RHAM THEORY

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Introduction. This note contains a method for constructing chain-homotopy operators suitable for the de Rham cohomology theory. In particular, it is proved that differentiably homotopic maps induce chain homotopic chain-mappings in the exterior algebra of differential forms (Formula 13 below; cf. pp. 80–81 of [1], where the same formula is obtained). This shows that the de Rham theory satisfies the "homotopy axiom" in the sense of S. Eilenberg and N. E. Steenrod (cf. [2]); hence the de Rham cohomology groups of a differentiably contractible manifold are trivial. This fundamental result is often referred to as the "Poincaré Lemma."

A simple generalization is given in the case of an almost product structure.

Almost complex and complex structures are investigated in §5; no genuine chain-homotopies are obtained, and in §6 is given an example which shows that $\bar{\partial}$-cohomology does not satisfy the homotopy axiom, even in the case of complex manifolds and analytic homotopies; this example is due to Professor K. Kodaira.

1. **Definitions and notations.** By "manifold" we mean "differentiable manifold of class C^∞," by "map," "map of class C^∞," etc.; and all notions such as tangent vector or differential form will be taken in their C^∞-sense. Tangent vectors will always be taken to have been defined by the C^∞-analogue of the definition given in §IV, Chap. II of [10].

If U is a manifold, we denote by $T^1(U)$ the tangent bundle, by $T(U)=\sum_{p=0}^{\infty} T^p(U)$ the bundle of exterior algebras of tangent vectors. Note that $T^0(U)=R=$the reals. By $\Phi(U)=\sum_{p=0}^{\infty}\Phi^p(U)$ we denote the exterior algebra of differential forms; for our purposes, the most convenient definition is

$$\Phi^p(U) = \mathrm{Hom}_{R(U)}[\times T^p(U), R(U)] \tag{1}$$

where $R(U)=\Phi^0(U)=R$-module of C^∞-maps $U\rightarrow R$, and $\times T^p(U)$ denotes the $R(U)$-module of cross-sections of $T^p(U)$. {If Λ is a commutative ring and A, B are Λ-modules, $\mathrm{Hom}_\Lambda(A, B)$ denotes the Λ-module of Λ-homomorphisms $A\rightarrow B$.}

If $v, v'\in \times T^p(U)$ are such that $v|V=v'|V$ ($v=v'$ "on V") where V

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is some open set of U, it is easy to see that $\phi v = \phi v'$ on V for $\phi \in \Phi^p(U)$. Hence the definition of $\Phi^p(U)$ is a "local" one; and $\phi \in \Phi^p(U)$ can be given by giving its values on germs of cross-section; a germ of cross-section at $x \in U$ is the equivalence class of all cross-sections which agree (pairwise) in some neighborhood of x.

If $\phi \in \Phi^{p+q}(U)$ and $v \in \times T^p(U)$ we define the contraction $v \rfloor \phi \in \Phi^q(U)$ by

$$(v \rfloor \phi)v' = \phi(v \wedge v')$$

where $v' \in \times T^q(U)$.

The exterior derivative $d: \Phi^p \to \Phi^{p+1}$ is given by the formula

$$(2) \qquad \begin{aligned}(d\phi)(v_1 \wedge \cdots \wedge v_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} v_i(\phi(v_1 \wedge \cdots \hat{\imath} \cdots \wedge v_{p+1})) \\ &+ \sum_{i<j} (-1)^{i+j+1} \phi([v_i, v_j] \wedge v_1 \wedge \cdots \hat{\imath} \cdots \hat{\jmath} \cdots \wedge v_{p+1})\end{aligned}$$

where the v_i are germs of $\times T^1(U)$, $\phi \in \Phi^p(U)$, $[v_i, v_j] = v_i v_j - v_j v_i$ and $\cdots \hat{\imath} \cdots$ denotes the omission of the term with index i. The following will be useful:

LEMMA 1. *The homomorphism d is uniquely characterized by*:
(i) *If* $\phi \in \Phi^0(U)$, $v \in \times T^1(U)$, $(d\phi)v = v\phi$,
(ii) *If* $\phi \in \Phi^0(U)$, $d^2\phi = 0$,
(iii) *If* $\phi \in \Phi^p(U)$, $\psi \in \Phi(U)$, $d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi$.

Since locally[2] $\Phi^1(U)$ is (isomorphic to) the Grassmann algebra generated by $\Phi^1(U)$ regarded as an $R(U)$-module, (ii) and (iii) imply (ii'): $d^2 = 0$.

If U, V are manifolds, and f: $U \to V$ is a map, we denote by $f_*: T(U) \to T(V)$ and $f^*: \Phi(V) \to \Phi(U)$ the corresponding induced maps.

If c is a differentiable (i.e., C^∞) p-chain in U and $\phi \in \Phi^p(U)$, we shall write $\phi \cdot c = \int_c \phi$. Stokes's theorem then takes the form $(d\phi) \cdot c = \phi \cdot bc$, where b denotes the boundary operator of the singular theory.

2. **Almost product structure.** We say that the manifold U has almost product structure (P, Q) if there are homomorphisms[3] P, Q: $T^1(U) \to T^1(U)$ such that $T^1(U) = PT^1(U) \oplus QT^1(U)$ (direct sum). Thus for $v \in T^1(U)$, $v = Pv + Qv$ and hence if $v_i \in T^1(U)$ $(i = 1, \cdots, p)$, then $v_1 \wedge \cdots \wedge v_p$ is a sum of terms each of which is the exterior

[2] More accurately: $\Phi(V)$ is the Grassmann algebra generated by $\Phi^1(V)$ for sufficiently small neighbourhoods $V \subset U$.

[3] For fibre bundles the fibres of which are modules, a homomorphism is a fibre-preserving map which, restricted to any fibre, is a homomorphism in the algebraic sense.

product of r vectors of type Pv_i by s vectors of type Qv_j where $r+s=p$; for given r, s such a term is called a vector of "type (r, s)"; and this process defines a unique projection operator

$$\prod_{r,s}: T^{r+s}(U) \to T^{r+s}(U)$$

onto the submodule of vectors of type (r, s).

We then define projection operators $\prod^*_{r,s}: \Phi^{r+s}(U) \to \Phi^{r+s}(U)$ by $\prod^*_{r,s}\phi = \phi \prod_{r,s}$. If $\phi = \prod^*_{r,s}\phi$ we say that ϕ is of type (r, s) (cf. [8]).

Let U, V be manifolds with almost product structures (P, Q), $(\overline{P}, \overline{Q})$ respectively. A map $F: U \to V$ is said to be of type (l, m) (in relation to these structures) if

$$\overline{\prod_{r+l,s+m}} F_* = F_* \prod_{r,s}.$$

A map of type (0, 0) is said to be *admissible*; the same definitions apply to any homomorphisms $T(U) \to T(V)$ or $\Phi(V) \to \Phi(U)$.

An examination of formula (2) shows that $d = d_2' + d_1' + d_1'' + d_2''$ where d_2', d_1', d_1'', d_2'' are of types $(2, -1)$, $(1, 0)$, $(0, 1)$, $(-1, 2)$ respectively. $d^2 = 0$ leads to the following identities:

$$\begin{aligned} d_2'^{\,2} &= d_2' d_1' + d_1' d_2' = d_2' d_1'' + (d_1')^2 + d_1'' d_2' \\ &= d_2' d_2'' + d_1' d_1'' + d_1'' d_1' + d_2'' d_2' \\ &= d_2'' d_1'' + d_1'' d_2'' = d_2'' d_1' + (d_1'')^2 + d_1' d_2' = (d_2'')^2 = 0. \end{aligned} \tag{3}$$

In analogy to Lemma 1, we now define the $R(U)$-homomorphism $d_P: \Phi^p(U) \to \Phi^{p+1}(U)$ by

(i) If $\phi \in \Phi^0(U)$ and $v \in \times T^1(U)$, then

$$(d_P\phi)v = (Pv)\phi.$$

(ii) If $\phi \in \Phi^0(U)$, $(d_P d + d d_P)\phi = 0$.

(iii) If $\phi \in \Phi^p(U)$ and $\psi \in \Phi(U)$,

$$d_P(\phi \wedge \psi) = d_P\phi \wedge \psi + (-1)^p \phi \wedge d_P\psi.$$

It easily follows that

(ii′) $d_P d + d d_P = 0$.

It is easily verified that $2d_2' + d_1' - d_2''$ satisfies these conditions; whence

$$d_P = 2d_2' + d_1' - d_2''. \tag{4}$$

Writing also

$$d_Q = 2d_2'' + d_1'' - d_2', \tag{5}$$

we see that $d=d_P+d_Q$ and, by symmetry, that d_Q is related to Q as d_P is to P.

Using (3), (4), we see that $d_P^2=d_1'^2+2(d_1'd_1''+d_1''d_1')+d_1''^2$. Hence, noting $d|\Phi^0(U)=d_1'+d_1''$ and appealing to Lemma 1, we have

LEMMA 2. *$d_P^2=0$ if and only if $d=d_1'+d_1''$; i.e., $d_2'=d_2''=0$; i.e., $d_P=d_1'$, $d_Q=d_1''$.*

It is not hard to prove that the conditions of Lemma 2 are equivalent to the "integrability" of the given almost product structure in which case we have a local product structure.

3. **The / operation.** Let U, V be manifolds. An obvious almost product structure is defined on $U\times V$ by regarding P, Q as the (natural) projection operators associated with the direct sum decomposition $T^1(U\times V)=T^1(U)\oplus T^1(V)$. We shall thus regard vector fields in U, V as lying, in an evident manner, in $U\times V$. It is clear that the conditions of Lemma 2 pertain; we write $d_U=d_P$, $d_V=d_Q$. d_U corresponds to "differentiation in U only."

Now, let $\phi\in\Phi^{r+s}(U\times V)$ and let c be a singular r-chain in U. Then (using a notation due to N. E. Steenrod, cf. [3]) we define $\phi/c\in\Phi^s(V)$ by

$$(\phi/c)(v)(y) = (-1)^{rs}[j_y^*(v \lrcorner \phi)]\cdot c \tag{6}$$

where $v\in\times T^s(V)$, $y\in V$ and $j_y: U\to U\times V$ is the map $x\to(x, y)$. Then, as is easily seen,

$$\begin{aligned}(-1)^{rs}d(\phi/c) &= (-1)^{r(s+1)}d_V\phi/c \\ &= (-1)^{r(s+1)}[d\phi/c - d_U\phi/c].\end{aligned} \tag{7}$$

Also, if $v'\in\times T^{s+1}(V)$ we have

$$\begin{aligned}(-1)^{r(s+1)}(d_U\phi/c)(v')(y) &= j_y^*(v' \lrcorner d_U\phi)\cdot c \\ = (-1)^{s+1}[dj_y^*(v' \lrcorner \phi)]\cdot c &= (-1)^{s+1}j_y^*(v' \lrcorner \phi)\cdot bc \\ &= (-1)^{r(s+1)}(\phi/bc)(v')(y).\end{aligned}$$

Hence

$$d\phi/c - \phi/bc = (-1)^r d(\phi/c) \tag{8}$$

(cf. 2.9 in [3]).

Now assume that V has almost product structure (P, Q) and that $U=U_P\times U_Q$. Define an almost product structure $(\overline{P}, \overline{Q})$ on $U\times V = U_P\times U_Q\times V$ by

$$\overline{P}T^1(U\times V) = T^1(U_P)\oplus PT^1(V),$$
$$\overline{Q}T^1(U\times V) = T^1(U_Q)\oplus QT^1(V).$$

In this situation, formula (8) splits up into various components. We discuss one special case, namely that when $c=c'\times x_Q$ where c' is an r-chain in U_P and x_Q is a point of U_Q regarded as a 0-chain. In this case the homomorphism $\phi\to\phi/c$ is of type $(-r, 0)$ in relation to the almost product structures $(\bar{P}, \bar{Q})$, (P, Q). By examining (8) in terms of its components we obtain:

$$
(9)\qquad \begin{aligned}
d_2'\phi/c &= (-1)^r d_2'(\phi/c),\\
d_1''\phi/c &= (-1)^r d_1''(\phi/c),\\
d_2''\phi/c &= (-1)^r d_2''(\phi/c),\\
d_1'\phi/c - \phi/bc &= (-1)^r d_1'(\phi/c)
\end{aligned}
$$

from which, using (4) and (5), we obtain

$$
(10)\qquad \begin{aligned}
d_{\bar{P}}\phi/c - \phi/bc &= (-1)^r d_P(\phi/c),\\
d_{\bar{Q}}\phi/c &= (-1)^r d_Q(\phi/c).
\end{aligned}
$$

4. **Chain homotopies.** Let us retain the notations of §3, let W be a third manifold and $F\colon U\times V\to W$ a map. We define $\lambda\colon\Phi(W)\to\Phi(V)$ by

$$
(11)\qquad \lambda\psi = (-1)^{r+1}(F^*\psi)/c
$$

for $\psi\in\Phi(W)$. Then, using (8) we get

$$
(12)\qquad (d\lambda + (-1)^{r+1}\lambda d)\psi = (F^*\psi)/bc.
$$

Now, consider the case when $c\colon I\to U$ is a singular 1-simplex and define $f_t\colon V\to W$ by $f_t(y)=F(c(t), y)$; then F represents a homotopy, and (12) becomes

$$
(13)\qquad d\lambda + \lambda d = f_1^* - f_0^*
$$

showing that differentiably homotopic maps induce chain-homotopic homomorphisms.

Next, consider the almost product structures introduced in the second part of §3, and assume that F is admissible (in relation to these structures). The homomorphism λ defined by (11) in terms of an r-chain c "in U_P" will be denoted by λ_P. Using (10), we get

$$
(14)\qquad \begin{aligned}
(d_P\lambda_P + (-1)^{r+1}\lambda_P d_P)\psi &= F^*\psi/bc,\\
d_Q\lambda_P + (-1)^{r+1}\lambda_P d_Q &= 0
\end{aligned}
$$

and finally, in analogy to (13),

$$
(15)\qquad d_P\lambda_P + \lambda_P d_P = f_1^* - f_0^*;
$$

in other words: A homotopy consistent with a given almost product structure induces chain-homotopies for the operator d_P; and similarly for d_Q.

5. **Almost complex structure** (cf. [**5**; **6**]). Let M be an m-manifold, and let $CT(M)=T(M)\otimes_R C$ where C are the complex numbers; and let $C(M)=C$-module of C^∞-maps $M\rightarrow C$. We define

$$C\Phi^p(M) = \mathrm{Hom}_{C(M)}[\times CT^p(M), C(M)] \tag{16}$$

and $C\Phi(M)=\sum_{p=0}^{\infty} C\Phi^p(M)$; cf. (1). We also define $d: C\Phi^p(M) \rightarrow C\Phi^{p+1}(M)$ by the formal analogue of (2); the definitions of f_*, f^* are similarly extended. It is clear that the whole "complex" theory is analogous to the "real" theory; also, $C(M)$ is naturally isomorphic to $R(M)\otimes_R C$, $C\Phi^p(M)$ to $\Phi^p(M)\otimes_R C$ and, under this isomorphism, d corresponds to $d\otimes 1$.

We say that M has a complex almost product structure if there are $C(M)$-homomorphisms P, $Q: CT^1(M)\rightarrow CT^1(M)$ such that $CT^1(M) = PCT^1(M)\oplus QCT^1(M)$, P, Q being projections. It is clear that the theory of almost product structures (§2 above) has an exact analogue in this situation: and we take over, without change, the definitions of $\prod_{r,s}$, $\prod_{r,s}^*$, "type (r, s)," $d=d_P+d_Q$, complete with Lemma 2.

We say that M has almost complex structure if it has complex almost product structure together with an isomorphism $k: CT^1(M) \rightarrow CT^1(M)$ such that $kPCT^1(M)=QCT^1(M)$, $kQCT^1(M)=PCT^1(M)$, $k^2=1$. Then k can be extended to $k: CT(M)\rightarrow CT(M)$ (and with a slight abuse of notation!) $k: C\Phi(M)\rightarrow C\Phi(M)$. We write $kv=\bar{v}$, $k\phi=\bar{\phi}$. In this case, in accordance with the usual notation, we write ∂, $\bar{\partial}$ for d_P, d_Q. If $\bar{\partial}^2=0$ (cf. Lemma 2) the given almost complex structure is called integrable (cf. [**5**]).

It is well known that if M has almost complex structure and n complex dimensions, then it can be assigned a Hermitian metric (cf. [**4**, p. 209]) and in terms of this a duality operator $*: C\Phi^p(M) \rightarrow C\Phi^{2n-p}(M)$ and a scalar product (ϕ, ψ) for ψ, $\phi\in C\Phi^p(M)$; cf [**1**; **5**; **7**; **8**]. These operations satisfy

$$\left(\prod_{r,s}^*\phi, \psi\right) = \left(\phi, \prod_{r,s}^*\psi\right),$$

$$*\prod_{r,s}^* = \prod_{n-r,n-s}^* *$$

and also, writing $\vartheta = - * \partial *$,

$$(\phi, \vartheta\psi) = (\bar{\partial}\phi, \psi)$$

if ϕ, ψ are forms with compact carriers (cf. [**5**]). We define

(17) $$\Delta = 2(\vartheta\bar{\partial} + \bar{\partial}\vartheta).$$

Now, let U be a subdomain (i.e., an open set) of M such that the closure of U in M is compact. By $\mathcal{L}$ denote the Hilbert space (in terms of the scalar product just introduced) of norm-finite differential forms on U and by $\mathcal{J}$ the space of forms $\phi \in C\Phi(M)$ such that $\bar{\partial}\phi = \vartheta\phi = 0$ and $\phi = 0$ outside U; then $\mathcal{J}$ can be regarded as a subspace of $\mathcal{L}$; we denote by $F: \mathcal{L} \to \mathcal{J}$ the associated projection operator. There exists a "Green's operator" $G: \mathcal{L} \to \mathcal{L}$ such that

(18) $$\Delta G\phi = \phi - F\phi$$

(cf. [**5**]).

Define $H, J: \mathcal{L} \to \mathcal{L}$ by

(19) $$H = 2\vartheta(\bar{\partial}G - G\bar{\partial}) + F, \qquad J = 2\vartheta G.$$

If the structure is complex, $\bar{\partial}^2 = 0$, $\vartheta^2 = 0$ and hence $\bar{\partial}\Delta = \Delta\bar{\partial}$, $\vartheta\Delta = \Delta\vartheta$; hence in this case $\Delta H = 0$. If U is a closed, compact manifold, $\bar{\partial}G - G\bar{\partial} = 0$ and $H = F$.

In the case of complex euclidean n-space, it is trivial that there exists a Green's operator G satisfying $\Delta G\phi = \phi$ and, if ϕ has a compact support, $\bar{\partial}G\phi = G\bar{\partial}\phi$. Hence, if we assume that U is an arbitrary subdomain of complex euclidean space, then $H\phi = 0$ provided that the support of ϕ is compact relative to U.

As is easily verified,

(20) $$\bar{\partial}J + J\bar{\partial} = I - H.$$

Let V be another almost complex manifold; give to $U \times V$ the natural induced almost complex structure; and by J_U, H_U, $\bar{\partial}_U$ denote the operators on $C\Phi(U \times V)$ associated with U. Then, if c is some singular r-chain in U, define $L: C\Phi(U \times V) \to C\Phi(V)$ by

(21) $$L\phi = J_U\phi/c.$$

It is easily seen that

(22) $$(-1)^{r+1}\bar{\partial}L + L\bar{\partial} = L\bar{\partial}_U$$

or, using (20),

(23) $$((-1)^{r+1}\bar{\partial}L + L\bar{\partial})\phi = (I - H_U - \bar{\partial}_U J_U)\phi/c.$$

Notice that, since there is no Stokes's formula in the geometrical sense for $\bar{\partial}$, no formula analogous to (8) can be obtained; similarly, no formulas analogous to (10) seem to exist, as singular chains cannot be closely related to almost complex structure.

In particular, if $r=0$, $\bar{\partial}_U J_U/c=0$ and (23) becomes

$$(-\bar{\partial}L + L\bar{\partial})\phi = (I - H_U)\phi/c. \tag{24}$$

Let W be a third almost complex manifold, $F: U\times V\to W$ a map such that $\bar{\partial}F^* = F^*\bar{\partial}$, let $c=u_1-u_0$ where u_0, $u_1\in U$, write $f_i(v) = F(u_i, v)$, and $\lambda = LF^*: C\Phi(W)\to C\Phi(V)$. Then, (24) gives

$$(-\bar{\partial}\lambda + \lambda\bar{\partial})\phi = (f_1^* - f_0^*)\phi - (H_U F^*\phi)_1 + (H_U F^*\phi)_0 \tag{25}$$

where $(H_U F^*\phi)_i = H_U F^*\phi | u_i \times V$.

If U is compact and connected, $H_U = F_U$ where F_U is the projection onto the space of forms satisfying $\vartheta_U\phi = \bar{\partial}_U\phi = 0$, and $\mathcal{J}_U^0$ (subspace of such forms of degree 0) is isomorphic to C; then $(H_U F^*\phi)_1 = (H_U F^*\phi)_0$ and (25) becomes

$$-\bar{\partial}\lambda + \lambda\bar{\partial} = f_1^* - f_0^*. \tag{26}$$

6. **An example** (cf. the Introduction). Let G be the multiplicative group consisting of matrices of the form

$$z = \begin{bmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{bmatrix}$$

where $z_i \in C$; let D be the (discrete) subgroup consisting of all z such that z_i are Gaussian integers. Then $V = G/D$ (i.e., the space of right cosets $z\cdot D$) is a homogeneous compact complex manifold (which was first considered by Iwasawa). It is easily seen that (in classical notation) the holomorphic 1-forms

$$w_1 = dz_1, \qquad w_2 = dz_2 - z_3 dz_1, \qquad w_3 = dz_3$$

are right invariant on G; they can hence be regarded as holomorphic 1-forms on V. Further, it is not hard to verify that w_1, w_2, w_3 generate the $\bar{\partial}$-homology group $H_{\bar{\partial}}^{1,0}(V)$ of forms of type (1, 0). Hence

$$\dim H_{\bar{\partial}}^{1,0}(V) = 3.$$

By the duality theorem of Kodaira-Serre (cf. [9; 11])

$$\dim H_{\bar{\partial}}^{2,3}(V) = 3.$$

It is easy to verify that

$$\begin{aligned} \psi_1 &= w_2 \wedge w_3 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3, \\ \psi_2 &= w_3 \wedge w_1 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3, \\ \psi_3 &= w_1 \wedge w_2 \wedge \bar{w}_1 \wedge \bar{w}_2 \wedge \bar{w}_3 \end{aligned}$$

represent linearly independent elements of $H_{\bar{\partial}}^{2,3}(V)$ and hence generate this group.

Now, every $t \in G$ induces the analytic homeomorphism $T_t: z \to t \cdot z$ of V onto itself; obviously each T_t is homotopic to the identity. We have

$$(T_t)^* w_1 = w_1,$$
$$(T_t)^* w_2 = w_2 - t_3 w_1 + t_1 w_3,$$
$$(T_t)^* w_3 = w_3$$

and hence

$$(T_t)^* \psi_1 = \psi_1 + t_3 \psi_2,$$
$$(T_t)^* \psi_2 = \psi_2,$$
$$(T_t)^* \psi_3 = \psi_3 - t_1 \psi_2$$

showing that there is *no* chain-homotopy.

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