ON INTERVAL RECURRENT SUMS OF INDEPENDENT RANDOM VARIABLES

FRANK SPITZER

1. **Introduction.** This note deals with infinite sequences $\{X_i\}$ of mutually independent identically distributed random variables. Their nth partial sums are denoted by $S_n = \sum_{i=1}^n X_i$. In all that follows each sequence $\{X_i\}$ is assumed to satisfy the further conditions:

(A) For some $1 \leqq \alpha \leqq 2$

$$\lim_{n\to\infty} \Pr \{S_n \leqq xn^{1/\alpha}\} = V_\alpha(x),$$

where $V_\alpha(x)$ is the distribution function of the symmetric stable law of order α (with the characteristic function $\exp\{-|\lambda|^\alpha\}$).

(B) The distribution for each X_i, $i=1, 2, \cdots$, is determined by a probability density $f(x)$, such that $f(x) \in L^p(-\infty, \infty)$ for some $p>1$.

Under the above conditions[1] it was shown by Kallianpur and Robbins [1] that the sequence of partial sums S_n is interval recurrent. This means that

$$\Pr\{|S_n - a| < \epsilon \text{ for infinitely many } n\} = 1 \tag{1}$$

for every a and every $\epsilon > 0$. If on the other hand (A) holds with $\alpha < 1$, then the probability in (1) is zero.

The results of this note take the form of two limit theorems of familiar type. They show that the recurrence behavior of the sequence $\{S_n\}$ depends only on the index α of the domain of attraction of the sequence. Condition (B) excludes from consideration all sequences of random variables of the lattice type for which the theorems below are clearly false. However it seems plausible that the sufficient conditions (A) and (B) may be replaced by far weaker conditions.

THEOREM 1.[2] *Let* $\{S_n\}$ *satisfy conditions* (A) *and* (B) *with exponent* $1 \leqq \alpha \leqq 2$. *Let*

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[1] In [1] it is required that $f(x) \in L^p(-\infty, \infty)$ for some $1 < p \leqq 2$. This condition is satisfied if $f(x) \in L^p(-\infty, \infty)$ for some $p > 1$ because $f(x) \in L^1(-\infty, \infty)$.

[2] Actually a far stronger result will be proved, the statement of which is contained in equation (12).

$$C(n, \alpha) = \begin{cases} \dfrac{1}{2} \alpha \sin \dfrac{\pi}{\alpha} n^{1/\alpha - 1} & \textit{for } 1 < \alpha \leqq 2, \\ \dfrac{1}{2} \pi (\log n)^{-1} & \textit{for } \alpha = 1, \end{cases}$$

and let $E_p(z)$ *denote the Mittag-Leffler function*

$$E_p(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(1 + pn)} \cdot$$

Then the limiting distribution of $\min_{1 \leqq k \leqq n} |S_k - a|$, *for any* $-\infty < a < \infty$, *normalized by the appropriate* $C(n, \alpha)$, *is given by*

$$\lim_{n\to\infty} \Pr \left\{ \min_{1 \leqq k \leqq n} |S_k - a| \leqq xC(n, \alpha) \right\} = \begin{cases} 0 & \text{for } x \leqq 0, \\ 1 - E_{1-1/\alpha}(-x) & \text{for } x \geqq 0. \end{cases}$$

Theorem 1 has a simple form when $\{S_n\}$ satisfies (A) and (B) with $\alpha = 1$ or $\alpha = 2$. When $\alpha = 1$, $x \geqq 0$,

$$\lim_{n\to\infty} \Pr \left\{ \min_{1 \leqq k \leqq n} |S_k - a| \leqq \frac{\pi}{2 \log n} x \right\} = \frac{x}{1 + x} \cdot$$

When $\alpha = 2$, $x \geqq 0$,

$$\lim_{n\to\infty} \Pr \left\{ \min_{1 \leqq k \leqq n} |S_k - a| \leqq \frac{x}{n^{1/2}} \right\} = 1 - \frac{2}{\pi^{1/2}} e^{x^2} \int_x^{\infty} e^{-t^2} dt.$$

Note that if (B) holds and if the random variables X_i have mean zero and finite variance σ^2, then the sequence $S_n = (2^{1/2}/\sigma) \sum_{i=1}^{n} X_i$ satisfies the conditions of Theorem 1 with $\alpha = 2$.

The strong law corresponding to Theorem 1 is

THEOREM 2. *With* $\{S_n\}$ *as in Theorem* 1, *and* $\{a_n\}$ *a nonincreasing positive sequence, the probability that* $|S_n - a| \leqq n^{1/\alpha} a_n$ *for infinitely many values of n is zero or one according as* $\sum_{n=1}^{\infty} a_n$ *converges or diverges.*

2. **Proof of Theorem** 1. For both theorems the following result due to Kallianpur and Robbins [1, Lemma 6.1] is basic.

"*Under the conditions* (A) *and* (B)[3] *the density* $f_n(x)$ *of* S_n *is, for large enough n, of the form*

$$f_n(x) = \frac{1}{\pi \alpha} \Gamma\left(\frac{1}{\alpha}\right) n^{-1/\alpha} + c_n g_n(x), \tag{2}$$

where $c_n = o(n^{-1/\alpha})$ *and* $|g_n(x)| < M(I)$ *for all x in any finite interval I.*"

[3] See footnote 1.

To prove Theorem 1 we define the events $E_k^n(x)$, $1 \leqq k \leqq n$, in the sample space of the sequence $\{X_i\}$ as

$$E_k^n(x) = \{w \mid |S_k - a| \leqq xC(n, \alpha)\}. \tag{3}$$

The random variables $I_k^n(\omega, x) = 1$ or 0 according as $\omega \in E_k^n$ or not are used to define the random variables

$$N_n(x) = N_n(\omega, x) = \sum_{k=1}^{n} I_k^n(\omega, x).$$

By a well-known combinatorial formula to be found in the book of Feller [2]

$$p_\nu(x, n) = \Pr\{N_n(x) = \nu\} = \sum_{k=\nu}^{n} (-1)^{k-\nu} \binom{k}{\nu} \rho_k(n, x), \tag{4}$$

where

$$\rho_k(n, x) = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \Pr\left[\bigcap_{s=1}^{k} E_{i_s}^n(x)\right], \qquad \rho_0(n, x) = 1. \tag{5}$$

Now it may be seen that Theorem 1 is equivalent to the statement

$$\lim_{n \to \infty} p_0(x, n) = E_{1-1/\alpha}(-x). \tag{6}$$

To prove (6) it must first be shown that

$$\rho_k(x) = \lim_{n \to \infty} \rho_k(n, x) = \frac{x^k}{\Gamma[1 + k(1 - 1/\alpha)]}. \tag{7}$$

If $f_n(x)$ is the density of S_n, and if $\Omega_k(n, x)$ is the region in Euclidean k-space defined by

$$\left| \sum_{i=1}^{s} x_i - a \right| \leqq xC(n, \alpha), \qquad s = 1, 2, \cdots, k,$$

then, according to (3) and (5),

$$\rho_k(n, x) = \sum_{1 \leqq i_1 < \cdots < i_k \leqq n} \int \cdots \int_{\Omega_k(n,x)} f_{i_1}(x_1) f_{i_2 - i_1}(x_2) \cdots f_{i_k - i_{k-1}}(x_k) dx_1 \cdots dx_k.$$

$$= \sum_{j_\nu \geqq 1, j_1 + \cdots + j_k \leqq n} \int \cdots \int_{\Omega_k(n,x)} \prod_{\nu=1}^{k} f_{j_\nu}(x_\nu) dx_\nu.$$

We may assume that (2) holds with $n \geqq N$ for our given sequence $\{X_i\}$. For the purpose of using (2) to prove (7) one can set

$$\sigma_k(n, x) \leqq \rho_k(n, x) \leqq \sigma_k(n, x) + \delta_k(n, x), \tag{8}$$

with

$$\sigma_k(n, x) = \sum_{j_\nu \geqq N, j_1 + \cdots + j_k \leqq n} \int_{\Omega_k(n,x)} \cdots \int \prod_{\nu=1}^{k} f_{j_\nu}(x_\nu) dx_\nu.$$

We note that for some $A_k > 0$, independent of n, each region $\Omega_k(n, x)$ is contained in the region

$$|x_i| \leqq A_k C(n, \alpha), \qquad i = 1, 2, \cdots, k.$$

Hence one may take

$$\prod_{\nu=1}^{k} \delta_\nu \quad \delta_k(n, x) = \sum_{1 \leqq \min_{1 \leqq \nu \leqq k} j_\nu < N, j_1 + j_2 + \cdots + j_k \leqq n} \prod_{\nu=1}^{k} \delta_\nu,$$

with

$$\delta_\nu = \int_{-A_k C(n,\alpha)}^{A_k C(n,\alpha)} f_{j_\nu}(x) dx \text{ or } 1 \text{ according as } j_\nu \geqq N \text{ or } j_\nu < N.$$

The volume of $\Omega_k(n, x)$ is $[2xC(n, \alpha)]^k$ and it follows from (2) that, as $n \to \infty$,

$$\sigma_k(n, x) = \left[2xC(n, \alpha) \cdot \frac{1}{\pi\alpha} \Gamma\left(\frac{1}{\alpha}\right)\right]^k \cdot [1 + o(1)] \sum_{j_\nu \geqq N, j_1 + \cdots + j_k \leqq n} [j_1 j_2 \cdots j_k]^{-1/\alpha},$$

where

$$2xC(n, \alpha) \cdot \frac{1}{\pi\alpha} \Gamma\left(\frac{1}{\alpha}\right) = \frac{x \cdot n^{1/\alpha - 1}}{\Gamma(1 - 1/\alpha)} \qquad \text{if } 1 < \alpha \leqq 2,$$

and

$$x/\log n \qquad \text{if } \alpha = 1.$$

It is clear that the limit of $\sigma_k(n, x)$ as $n \to \infty$ does not depend on the value of N, and a calculation which is similar to those carried out in [1] and [3] yields

$$\lim_{n \to \infty} \sigma_k(n, x) = \frac{x^k}{\Gamma[1 + k(1 - 1/\alpha)]}.$$

Straightforward calculation also shows that $\delta_k(n, x) \to 0$ as $n \to \infty$, and in view of (8) the proof of (7) is now complete.

To prove (6), and thus Theorem 1, it would suffice to show directly that

$$\lim_{n\to\infty} p_0(x,n) = \lim_{n\to\infty} \sum_{k=0}^{n} (-1)^k \rho_k(n,x)$$
$$= \sum_{k=0}^{\infty} (-1)^k \lim_{n\to\infty} \rho_k(n,x) = E_{1-1/\alpha}(-x).$$

However, it seems impossible to justify the interchange of limits. The method used instead is longer but has the advantage of yielding a stronger result.

Equation (4) implies that the kth moment of $N_n(x)$ is

$$E\{[N_n(x)]^k\} = \sum_{\nu=0}^{n} p_\nu(x,n)\nu^k = \sum_{\nu=0}^{n} \rho_\nu(n,x)V_\nu(k),$$

where

$$V_\nu(k) = \sum_{\mu=0}^{\nu} \binom{\nu}{\mu} (-1)^{\nu-\mu}\mu^k. \tag{9}$$

Thus $(1/k!)V_\nu(k)$ is the coefficient of x^k in the power series expansion of $(e^x-1)^\nu$. Hence $V_\nu(k)=0$ for $\nu>k$, so that $E\{[N_n(x)]^k\}$ is a finite sum of $k+1$ terms for any value of n. Hence

$$m_k = \lim_{n\to\infty} E\{[N_n(x)]^k\} = \sum_{\nu=0}^{k} \rho_\nu(x)V_\nu(k), \tag{10}$$

with $\rho_\nu(x)$ given in equation (7). Let $\phi_n(\lambda)$ be the characteristic function of $N_n(x)$, and define, using (7), (9), and (10),

$$\begin{aligned}\phi(\lambda) &= \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} m_k = \sum_{k=0}^{\infty} \frac{(i\lambda)^k}{k!} \sum_{\nu=0}^{k} \rho_\nu(x)V_\nu(k) \\ &= \sum_{\nu=0}^{\infty} \rho_\nu(x)(e^{i\lambda}-1)^\nu = E_{1-1/\alpha}(xe^{i\lambda}-x).\end{aligned} \tag{11}$$

$\phi(\lambda)$ is seen to be analytic in a neighborhood of the origin for every $x>0$ and for every $1\leqq\alpha\leqq 2$. Hence the moment problem defined by equation (10) has a unique solution, and the sequence $\phi_n(\lambda)$ converges to the characteristic function $\phi(\lambda)$. Since $\phi(\lambda)$ is periodic with period 2π for real λ, it is the characteristic function of a distribution function which is constant except for jumps $p_k(x)$ at the points $k=0, 1, \cdots$. But as $\phi_n(\lambda)\to\phi(\lambda)$ we must have

$$p_k(x) = \lim_{n\to\infty} p_k(x,n).$$

Moreover, for $|t|\leqq 1$, equation (11) implies

$$\sum_{k=0}^{\infty} p_k(x)t^k = \lim_{n\to\infty} E\left[t^{N_n(x)}\right] = E_{1-1/\alpha}(xt - x). \tag{12}$$

Letting t tend to zero in (12) we have

$$p_0(x) = \lim_{n\to\infty} \Pr\{N_n(x) = 0\} = E_{1-1/\alpha}(-x),$$

so that Theorem 1 has been established as a special case of the result of equation (12).

3. **Proof of Theorem 2.** We shall assume that $a=0$, the method of proof being the same for any value of a. Let $E_n = \{\omega \mid S_n \in I_n\}$ where I_n is the interval $(-n^{1/\alpha}a_n, n^{1/\alpha}a_n)$. First assume that $\sum_{n=1}^{\infty} a_n < \infty$. Then $n^{1/\alpha}a_n \to 0$ as $n\to\infty$, and one obtains from (2) that

$$\begin{aligned} \Pr\{E_n\} &= \int_{I_n} f_n(x)dx = 2n^{1/\alpha}a_n\left[\frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right)n^{-1/\alpha} + o(n^{-1/\alpha})\right] \\ &= Aa_n + o(a_n) \text{ as } n\to\infty, \text{ with } A = \frac{2}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right). \end{aligned} \tag{13}$$

Hence $\sum_{n=1}^{\infty} \Pr\{E_n\} < \infty$, and by the Borel-Cantelli lemma

$$\Pr\{E_n \text{ for infinitely many } n\} = 0,$$

which proves the first part of Theorem 2.

To prove the second part of Theorem 2 one has to assume that $\sum_{n=1}^{\infty} a_n = \infty$. In fact, it will clearly be sufficient to consider only the case when, in addition to $\sum_{n=1}^{\infty} a_n = \infty$,

$$a_n \leqq [n \log n]^{-1} \qquad \text{for large enough values of } n. \tag{14}$$

Just as before, one shows that $\sum_{n=1}^{\infty} \Pr\{E_n\} = \infty$. The events E_n are not independent, but to conclude that

$$\Pr\{E_n \text{ for infinitely many } n\} = 1$$

it will suffice, according to Chung and Erdös [**4**], to verify that the sequence $\{E_n\}$ satisfies the following three conditions.

(I) For any positive integers n and h, $n \geqq h$, there exist positive functions $c(h)$ and $H(n, h) > h$ such that for every $k > H(n, h)$

$$\Pr\{E_k \mid E_h' \cap E_{h+1}' \cap \cdots \cap E_n'\} > c(h) \Pr\{E_k\}.$$

(Primes denote the complements.)

(II) A set of events $\{E_{n_i}\}$, $i = 1, 2, \cdots, s(n)$ can be found for each value of n with the property

$$\sum_{i=1}^{s(n)} \Pr\{E_{n_i} \mid E_n\} < c_1,$$

where c_1 is a positive constant independent of n.

(III) There is a constant $c_2>0$, such that

$$\Pr\left\{E_k \mid E_n\right\} < c_2 \Pr\left\{E_k\right\} \text{ for all } n,$$

whenever $k>n$ and E_k is not in the set $\{E_{n_i}\}$ chosen in (II) to correspond to E_n.

With E_k and the interval I_k as defined above,

$$\Pr\left\{E_k \mid E_h' \cap \cdots \cap E_n'\right\} = \int_{I_k}\int_{-\infty}^{\infty} f_{k-n}(x-y)\,dG_{n,h}(y)\,dx, \tag{15}$$

where $G_{n,h}(x)$ is the conditional probability distribution

$$G_{n,h}(x) = \Pr\left\{S_n \leqq x \mid E_h' \cap \cdots \cap E_n'\right\}.$$

Let $I=I(n,h)$ be a bounded interval with the property that

$$\int_I dG_{n,h}(x) = \frac{1}{2}\cdot$$

Then equation (2) applied to (15) implies

$$\begin{aligned}\Pr\left\{E_k \mid E_h' \cap \cdots \cap E_n'\right\} &\geqq \int_{I_k}\int_I f_{k-n}(x-y)\,dG_{n,h}(y)\,dx \\ &= \frac{1}{2}Aa_k + o(a_k),\end{aligned} \tag{16}$$

where A is the constant defined in (13). Combining (16) and (13) one obtains

$$\Pr\left\{E_k \mid E_h' \cap \cdots \cap E_n'\right\} \geqq \Pr\left\{E_k\right\}\cdot[1/2 + o(1)], \text{ as } k\to\infty, \tag{17}$$

so that condition (I) is satisfied with $c(h)=1/3$.

To verify condition (II), let the set $\{E_{n_i}\}$ which corresponds to E_n consist of the events $E_{n+1}, E_{n+2}, \cdots, E_{2n}$. To show that the sequence of sums $\sum_{k=1}^n \Pr\left\{E_{n+k} \mid E_n\right\}$ is bounded, one can write

$$\Pr\left\{E_{n+k} \mid E_n\right\} = \int_{I_{n+k}}\int_{I_n} f_k(x-y)\,dH_n(y)\,dx, \tag{18}$$

where

$$H_n(x) = \Pr\left\{S_n \leqq x \mid S_n \in I_n\right\}.$$

Equation (2) applied to (18) now shows that, as $k\to\infty$,

$$\Pr\left\{E_{n+k} \mid E_n\right\} = A(n+k)^{1/\alpha}k^{-1/\alpha}a_{n+k} + (n+k)^{1/\alpha}a_{n+k}o(k^{-1/\alpha}). \tag{19}$$

Using (14), the boundedness of the sequence $\sum_{k=1}^n \Pr\left\{E_{n+k} \mid E_n\right\}$ is a consequence of the existence of the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{(n+k)^{1/\alpha-1}k^{-1/\alpha}}{\log(n+k)} < \infty, \qquad \text{for } 1 \leqq \alpha \leqq 2.$$

Finally, to verify condition (III), we must show that the ratios

$$\frac{\Pr\{E_n \mid E_k\}}{\Pr\{E_n\}}$$

are bounded for all $n > 2k$. For large enough k, (19) and (13) combined yield

$$\frac{\Pr\{E_n \mid E_k\}}{\Pr\{E_n\}} = \left(\frac{k}{n}\right)^{-1/\alpha} + o(1) \leqq 2^{-1/\alpha} + o(1),$$

so that (III) holds, and the proof of Theorem 2 is complete.

Bibliography

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California Institute of Technology