A NOTE ON TWO GENERATING FUNCTIONS FOR LEGENDRE FUNCTIONS

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It has recently been shown by Bloch [1] that for Legendre functions not on the cut

$$(1) \quad e^{tz} I_m\left[t(z^2-1)^{1/2}\right] = \sum_{n=0}^{\infty} P^m_{m+n}(z) \frac{t^{m+n}}{(2m+n)!}, \qquad m = 0, 1, 2, \cdots,$$

where

$$I_m(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{m+2n}}{n!(m+n)!}, \qquad P^m_n(z) = (z^2-1)^{m/2} d^m P_n(z)/dz^m.$$

It is the purpose of this note to show how (1) and the result (2) below may be derived using methods of Truesdell [2] and thus to correct the results (58) and (66) on pages 101 and 105 of [2].[1]

If one applies Theorem 14.4 of [2] to the function

$$F(z,\alpha) = \frac{\cos \pi(\alpha - m)}{\Gamma(1-\alpha+2m)} (z^2-1)^{-(\alpha-m)/2} P^m_{m-\alpha}\left[\frac{-z}{(z^2-1)^{1/2}}\right],$$

the changes of variable $x = -z(z^2-1)^{-1/2}$, $yz = xt$ lead after simplifications to the generating function relationship (1). Application of Theorem 14.7 of [2] to the function

$$F(z,\alpha) = \Gamma(\alpha - b + 1)(z^2-1)^{-(\alpha+1)/2} P^b_\alpha\left[\frac{-z}{(z^2-1)^{1/2}}\right]$$

gives the formula

$$\sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n-b+1)}{n!\Gamma(\alpha+n+1)} (z^2-1)^{-(\alpha+n+1)/2} P^b_{\alpha+n}\left[\frac{-z}{(z^2-1)^{1/2}}\right] y^n$$
$$= \frac{\Gamma(\alpha-b+1)}{2\pi i} \int_C e^w w^{-\alpha} \left[(z+y/w)^2 - 1\right]^{-(\alpha+1)/2}$$
$$\cdot P^b_\alpha\left[\frac{-z-y/w}{\{(z+y/w)^2-1\}^{1/2}}\right] dw,$$

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[1] Truesdell's results are incorrect because of computational error. The incorrect formula 19.10(17) of [3] should also be replaced by (1).

where the contour of integration extends from $-\infty$ counterclockwise about the points $y/(1-z)$ and $-y/(1+z)$ and back to $-\infty$. Upon making the substitutions $\alpha=b=m$, $m=0, 1, 2, \cdots, x=-z(z^2-1)^{-1/2}$, $y(z^2-1)^{-1/2}=t$, and $w=ts$, one finds after simplifications that for the Legendre functions away from the cut

$$\frac{(2m-1)!!(x^2-1)^{m/2}}{2\pi i}\int_C e^{st}(1-2xs+s^2)^{-m-1/2}s^m ds = \sum_{n=0}^{\infty} P_{n+m}^{m}(x)\frac{t^{m+n}}{(m+n)!}. \tag{2}$$

The variables y and t are real. The contour C now extends from $-\infty$ counterclockwise around the zeros of $(1-2xs+s^2)$ and back to $-\infty$.

References

1. E. L. Bloch, *On an expansion of Bessel functions in a series of Legendre functions*, Akad. Nauk SSSR. Prikl. Mat. Meh. vol. 18 (1954) pp. 745–748.

2. C. Truesdell, *A unified theory of special functions*, Princeton University Press, 1948.

3. The Bateman Manuscript Project, *Higher transcendental functions*, vol. 3, McGraw-Hill, 1955.

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