

THE EXISTENCE OF A VECTOR OF WEIGHT 0 IN IRREDUCIBLE LIE GROUPS WITHOUT CENTRE

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R. Bott and H. Samelson raised the question (communicated to me by W. T. van Est) whether it is true that every irreducible linear, semi-simple compact Lie group with trivial centre possesses a vector of weight 0. In the following I shall give a proof of this conjecture and of its converse. In the meantime R. Bott wrote me that Harish-Chandra also proved this conjecture. His proof is quite different from the present one.

Let G be an irreducible linear, semi-simple compact Lie group.

THEOREM. *The centre of G is trivial if and only if G possesses a vector of weight 0.*

Let H be a Cartan subgroup of G . Since G is semi-simple and compact every element of G is conjugate to an element of H .¹ Therefore the centre C of G is contained in H .

Furthermore every element of H is generated by an infinitesimal element of H .

The centre C of G consists of those elements of G which in the adjoint representation $\text{ad } G$ are represented by 1.

Therefore if h is an infinitesimal element of H , then $\exp h \in C$ if and only if

$$\alpha(\text{ad } h) \equiv 0 \pmod{2\pi i}$$

for each root form α .

This central element is trivial, if and only if

$$\omega(\text{ad } h) \equiv 0 \pmod{2\pi i}$$

for each weight ω of G .

Therefore there is equivalence between the two assertions:

- (1) C is trivial.
- (2) $\alpha(\text{ad } h) \equiv 0 \pmod{2\pi i}$ for all root forms implies $\omega(\text{ad } h) \equiv 0 \pmod{2\pi i}$ for all weights.

The second assertion may also be stated in the form:

- (3) Every weight is a linear combination of root forms with integer coefficients.

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¹ E. Cartan, *Annali di Matematica Pura ed Applicata* (4) vol. 4 p. 214.

We must now show that (3) is equivalent to:

(4) There is a weight 0.

Proof (3)→(4): We choose the weight ω and the representation

$$\omega = \sum_{\nu=1}^p \alpha_{\nu}$$

by a system of root forms α_{ν} , so that p is minimal. If $p > 0$, then under the Weyl normalization ($N_{\alpha} = -1$) of the Killing-Cartan quadratic form written as an inner product, we have

$$(\alpha, \alpha) < 0 \quad \text{and} \quad (\omega, \omega) < 0$$

for all root forms and weights.

$$(\omega, \omega) = \sum_{\nu} (\omega, \alpha_{\nu}).$$

Therefore

$$(\omega, \alpha_{\nu}) < 0$$

for at least one ν , and

$$2 \frac{(\omega, \alpha_{\nu})}{(\alpha_{\nu}, \alpha_{\nu})} \cong 1.$$

This implies that $\omega - \alpha_{\nu}$ is also a weight, in contradiction to the minimality property of p . Therefore p must be 0, and 0 must be a weight.

(4)→(3): Because of the irreducibility of G every weight can be derived from the highest weight by successive subtraction of root forms. Therefore if 0 is a weight, then the highest weight is a sum of root forms and then every other weight has the same property.

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