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A THEOREM OF ÉLIE CARTAN

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André Weil [1] and Hopf and Samelson [2] have given a topological proof of the following theorem of Élie Cartan.

Two maximal Abelian subgroups of a compact connected Lie group $\mathcal{G}$ are conjugate within $\mathcal{G}$.

I present a simple metric proof.

LEMMA. *If x and y are elements of the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$ then $[x, A_\sigma y]$ vanishes for some inner automorphism A_σ of $\mathcal{G}$.*

PROOF. Because $\mathcal{G}$ is compact one can define on $\mathfrak{g}$ a nonsingular bilinear form (u, v) which is invariant: $([u, v], w)+(v, [u, w])\equiv 0$. We choose ϵ in $\mathcal{G}$ so that $(x, A_\sigma y)$ attains its minimum for $\sigma=\epsilon$; without loss of generality we may assume ϵ to be the neutral element of $\mathcal{G}$, and then $A_\epsilon y=y$. If now z is any element of $\mathfrak{g}$ the function $(x, A_{\exp (tz)}\, y)$ has a minimum for $t=0$, so that its derivative vanishes there. Thus, keeping in mind that

$$\frac{d}{dt} A_{\exp (tz)} y \bigg|_{t=0} = [z, y],$$

we have $(x, [z, y])=0$. From this equation and from the invariance of the bilinear form it follows that $([x, y], z)=0$ for all z; this can happen only if $[x, y]$ vanishes, for the bilinear form is nondegenerate.

Before proving Cartan's theorem I recall some well-known facts: A maximal Abelian subgroup $\mathcal{H}$ of $\mathcal{G}$ is a torus group; there is an element x in the Lie algebra $\mathfrak{h}$ of $\mathcal{H}$ such that the one parameter group $\exp tx$ is dense in $\mathcal{H}$; if y belongs to $\mathfrak{g}$ and $[x, y]=0$, then y must lie in $\mathfrak{h}$.

Matters being so, let $\mathcal{H}'$ be a second maximal Abelian subgroup of $\mathcal{G}$ and x' an element of its Lie algebra bearing the same relation

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to $\mathcal{H}'$ as x does to $\mathcal{H}$. Now choose σ in $\mathcal{G}$ so that $[x, A_\sigma x']$ vanishes. Then $A_\sigma x'$ lies in $\mathfrak{h}$; consequently $A_\sigma(\exp tx') \equiv \exp (tA_\sigma x')$ lies in $\mathcal{H}$ for every t. So $\mathcal{H}$, being closed, includes the closure $A_\sigma(\mathcal{H}')$ of the one-parameter group $A_\sigma(\exp tx')$. Finally $A_\sigma(\mathcal{H}') = \mathcal{H}$, because both are maximal Abelian subgroups of $\mathcal{G}$.

Since every element of $\mathcal{G}$ can be written as $\exp y$, the argument shows that every element of $\mathcal{G}$ can be moved into $\mathcal{H}$ by an inner automorphism of $\mathcal{G}$.

The referee has pointed out that the argument of the lemma above is very like one used by R. Bott [**3**] in another context.

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