THE DERIVATIVE OF A MEROMORPHIC FUNCTION

J. CLUNIE

1. It has been shown by Valiron [2] and Whittaker [3] that the derivative of a meromorphic function of finite order is of the same order as the function itself. This result, as pointed out by Whittaker, is equivalent to the following.

THEOREM. *If* $f(z)$ *and* $g(z)$ *are two integral functions of orders* ρ_1 *and* ρ_2 *with* $\rho_1 > \rho_2$, *then* $f'(z)g(z) - f(z)g'(z)$ *is of order* ρ_1.

The proofs given by Valiron and Whittaker depend on meromorphic function theory, but in this paper I shall give a proof of the above theorem which depends entirely on integral function theory.

2. A number of lemmas are required and no proof will be given for the first of these as it is already well known.

LEMMA 1 [1, p. 102]. *Except for an exceptional set of intervals within which the variation of* $\log r$ *is finite*

$$zf'(z) = Nf(z)\{1 + o(1)\}, \qquad |z| = r,$$

where $|f(z)| = M(r, f)$ *and* $N = N(r, f)$.

LEMMA 2. *There is an infinite sequence* $\{\lambda_i\}_1^\infty$ *such that if* $\lambda_i \leqq r \leqq \lambda_i^\alpha$, $\alpha = (\rho_1 - \epsilon/2)(\rho_1 - \epsilon)^{-1}$, *then*

$$\log N(r, f) \geqq (\rho_1 - \epsilon) \log r.$$

From the result [1, p. 33]

$$\limsup_{r\to\infty} \{\log N(r, f)\}/\log r = \rho_1$$

it follows that there is an infinite sequence $\{\lambda_i\}$ such that

$$\{\log N(\lambda_i, f)\}/\log \lambda_i \geqq \rho_1 - \epsilon/2.$$

Since $N(r, f)$ is a nondecreasing function of r this means that

$$\{\log N(r, f)\}/\log r \geqq \rho_1 - \epsilon$$

provided $(\rho_1 - \epsilon/2) \log \lambda_i \geqq (\rho_1 - \epsilon) \log r$, which gives the lemma.

Received by the editors April 5, 1955.

LEMMA 3. *Except when r lies in a set of intervals of total finite length*

$$|f'(z)/f(z)| < r^{\rho_1+\epsilon}, \qquad |z| = r, \epsilon > 0.$$

If p is the genus of $f(z)$ then

$$f'(z)/f(z) = \rho(z) + \sum_1^\infty z^p\{(z - a_n)a_n^p\}^{-1}$$

where $\{a_n\}$ is the sequence of zeros of $f(z)$ and $\rho(z)$ is a polynomial whose degree does not exceed $\rho_1 - 1$. Hence

$$|f'(z)/f(z)| \leqq O(r^{\rho_1-1}) + r^p \sum_1^\infty \{|r - r_n| r_n^p\}^{-1}.$$

Given σ and $k > 1$ we suppose $\sigma k^{-1/2} \leqq r \leqq \sigma k^{1/2}$ and consider the above sum in three parts $\sum_1$, $\sum_2$, $\sum_3$ such that $r_n < \sigma k^{-1}$, $\sigma k^{-1} \leqq r_n \leqq \sigma k$, $r_n > \sigma k$ respectively. Using the fact that $p+1$ exceeds the exponent of convergence of the zeros, the sums $\sum_1$ and $\sum_3$ can be shown to satisfy

$$\sum_1 = O(r^p), \qquad \sum_3 = O(r^p).$$

Integrating $\sum_2$ with respect to r over $(\sigma k^{-1/2}, \sigma k^{1/2})$, excluding intervals of length 2η centered on each r_n, where $\eta = (\sigma k^{-1/2})^{-h}$, $h > \rho_1$, we get

$$\int \sum_2 dr = O(\sigma^{\rho_1+\delta} \log \sigma)$$

provided, given $\delta > 0$, we choose σ sufficiently large. The range of integration is of length not less than $(k^{1/2} - k^{-1/2})\sigma - 2n(k\sigma)(\sigma k^{-1/2})^{-h}$, which is $(k^{1/2} - k^{-1/2})\sigma - O(1)$. This means that $\sum_2 < r^{\rho_1+2\delta}$ except in a set of intervals whose lengths in sum do not exceed $O(\sigma^{-\delta} \log \sigma)$. Combining these results gives the lemma.

3. The proof of the theorem will now be given. From the lemmas it follows that we can find an infinite sequence of r such that, with $|z| = r$,

$$\log N(\sigma, f) \geqq (\rho_1 - \epsilon) \log \sigma, \quad r^\alpha \leqq \sigma \leqq r, \quad \alpha = (\rho_1 - \epsilon)(\rho_1 - \epsilon/2)^{-1},$$

$$zf'(z)/f(z) = N(r, f)\{1 + o(1)\}, \quad |f(z)| = M(r, f),$$

$$|g'(z)/g(z)| < r^{\rho_2+\epsilon}.$$

Also [1, p. 32]

$$\begin{aligned}\log M(r, f) &\sim \log \mu(r, f)\\ &= K + \int_1^r x^{-1}N(x, f)dx\\ &> \int_{r\alpha}^r x^{-1}N(x, f)dx\\ &> r^{\rho_1-\epsilon}\end{aligned}$$

if r is large enough. Finally

$$\begin{aligned}|f'(z)g(z) - f(z)g'(z)| &> |f(z)g(z)| \{ |f'(z)/f(z)| - |g'(z)/g(z)| \}\\ &> \exp (r^{\rho_1-\epsilon} - r^{\rho_2+\epsilon})(r^{\rho_1-1-\epsilon} - r^{\rho_2+\epsilon})\end{aligned}$$

where use has been made of a result of Borel [1, p. 57]. Thus the theorem follows if $\rho_1 - 1 > \rho_2$. When this is not the case we choose an integer n so that $n\rho_1 - 1 > n\rho_2$ and define $F(z) = f(z^n)$, $G(z) = g(z^n)$. Then

$$F'(z)G(z) - F(z)G'(z) = nz^{n-1}\{f(z^n)g(z^n) - f(z^n)g'(z^n)\}$$

and by the previous result $F'(z)G(z) - F(z)G'(z)$ is of order $n\rho_1$. Hence $f'(z)g(z) - f(z)g'(z)$ is of order ρ_1 which proves the theorem in this case also.

4. As pointed out by Whittaker the above result is the best possible, as can be seen by taking $f(z) = \cos z$ and $g(z) = \sin z$. However, it would be of interest to know if the theorem was true for functions of the same order if the type of $g(z)$ were less than that of $f(z)$.

It would also be of interest to know what can be said about the order of $f'(z)g(z) + f(z)h(z)$ when $g(z)$ and $h(z)$ have orders less than that of $f(z)$.

References

1. G. Valiron, *Lectures on the general theory of integral functions*, Toulouse, 1923.
2. ———, Acta Math. vol. 47 (1925) p. 129.
3. J. M. Whittaker, J. London Math. Soc. vol. 2 (1936) pp. 82–87.

University College of North Staffordshire