A FAMILY OF BEST TWIN CONVERGENCE REGIONS FOR CONTINUED FRACTIONS[1]

VIKRAMADITYA SINGH AND W. J. THRON

Two regions E_1 and E_2 in the complex plane are called *twin convergence regions for the continued fraction*

$$(1) \qquad 1 + \frac{c_1^2}{1} + \frac{c_2^2}{1} + \cdots,$$

if the fact that $c_{2n-1} \in E_1$, $c_{2n} \in E_2$, for all $n \geqq 1$, is sufficient to insure the convergence of (1). Twin convergence regions E_1 and E_2 are called *best* twin convergence regions, if there do not exist regions $E_1' \supset E_1$ and $E_2' \supset E_2$, where at least one of the equalities $E_1' = E_1$, $E_2' = E_2$ fails to hold, which are twin convergence regions for the continued fraction (1).

Some time ago one of the authors [1] proved that the continued fraction (1) converges if

$$(2) \qquad |c_{2n-1}| \leqq \rho, \qquad |c_{2n} \pm i| \geqq \rho + \epsilon, \qquad n \geqq 1,$$

where $0 < \rho < 1$ and ϵ is an arbitrary positive quantity. It was also shown in [1] that if the ϵ could be removed in (2) a set of best twin convergence regions would be obtained. It is the purpose of this paper to show that the ϵ can indeed be removed.

Denote by $t_n(z)$ the linear fractional transformation

$$1 + c_n^2/z.$$

It was shown in [1] that if

$$(3) \qquad |c_{2n-1}| \leqq \rho, \qquad |c_{2n} \pm i| \geqq \rho \qquad \text{for all } n \geqq 1,$$

then

$$|1 - t_{2n-1}(z)| \leqq \rho \qquad \text{for all } |z| \geqq \rho,$$

and

$$|t_{2n}(z)| \geqq \rho \qquad \text{for all } |z - 1| \leqq \rho.$$

Set

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$$T_n(z) = t_1(t_2(\cdots t_n(z)\cdots)),$$

and denote by Z_1 and Z_2 the sets defined by $|z| \geqq \rho$, $|z-1| \leqq \rho$, respectively. The sets

$$K_{2n} = T_{2n}(Z_2), \qquad K_{2n-1} = T_{2n-1}(Z_1)$$

are circular regions and $|K_n - 1| \leqq \rho$ for all $n \geqq 1$. Also $K_{n-1} \supset K_n$ for all $n \geqq 2$. Thus we have the well known (see for example [2, pp. 70–79]) "nest of circles." Finally, since $1 \in Z_1$, $1 \in Z_2$ and since the nth approximant of the continued fraction (1) is

$$\frac{A_n}{B_n} = T_n(1),$$

we have for all $n \geqq m$, $A_n/B_n \in K_m$.

Let R_n be the radius of K_n. Then $\{R_n\}$ forms a nonincreasing sequence. Hence if it can be shown that a subsequence of $\{R_n\}$ converges to zero it will follow that the continued fraction converges. Our proof consists in showing that $\lim R_{2n} = 0$.

We next observe that a consequence of the fundamental recurrence relation

$$B_n = B_{n-1} + c_n^2 B_{n-2}, \tag{4}$$

is that

$$\frac{B_n}{B_{n-1}} = 1 + \frac{c_n^2}{1} + \frac{c_{n-1}^2}{1} + \cdots + \frac{c_2^2}{1}.$$

Thus

$$\frac{B_n}{B_{n-1}} = t_n(\cdots t_2(1)\cdots).$$

From this it follows that

$$\left|\frac{B_{2n-1}}{B_{2n-2}} - 1\right| \leqq \rho, \qquad \left|\frac{B_{2n}}{B_{2n-1}}\right| \geqq \rho. \tag{5}$$

The inequality (5) allows us to conclude that, if condition (3) holds, no $B_n = 0$. Now set

$$m_n = \min\left|\frac{B_{2n}}{B_{2n-1}}\right|, \text{ and } k_n = \max\left|\frac{B_{2n-1}}{B_{2n-2}} - 1\right|.$$

Condition (5) insures that, for all n, $m_n \geqq \rho$ and $k_n \leqq \rho$. It is, however, desirable to obtain sharper estimates, depending on n, for these quantities. This is done in the following:

LEMMA 1. *If the elements c_n satisfy condition* (3) *then*

$$m_n = \rho(1 + (1 - \rho^2)/(n - \rho(n - 2))). \tag{6}$$

Since

$$\left|\frac{B_{2n-1}}{B_{2n-2}} - 1\right| = \frac{|c_{2n-1}^2|}{|B_{2n-2}/B_{2n-3}|},$$

we have

$$k_n = \rho^2/m_{n-1}. \tag{7}$$

Equation (6) is readily verified for $n=1$ since

$$m_1 = \min |1 + c_2^2| = \rho(2 - \rho).$$

Now

$$m_n = \min\left|1 + \frac{c_{2n}^2}{B_{2n-1}/B_{2n-2}}\right|,$$

and one easily sees that the minimum is attained when c_{2n} is of the form $i+\rho e^{i\theta}$ (we can assume that all c_n are in the upper half-plane) and B_{2n-1}/B_{2n-2} is of the form $1+k_n e^{i\phi}$. Thus

$$m_n = \min\left|1 + \frac{(i + \rho e^{i\theta})^2}{1 + k_n e^{i\phi}}\right|, \qquad 0 \leqq \theta \leqq 2\pi,\ 0 \leqq \phi \leqq 2\pi.$$

Now

$$\left|1 + \frac{(i + \rho e^{i\theta})^2}{1 + k_n e^{i\phi}}\right| = \left|1 - \frac{(1 - \rho i e^{i\theta})^2}{1 - k_n^2}(1 + k_n e^{i\phi'})\right|,$$

where ϕ' depends on k_n, ρ, and ϕ, but ranges from 0 to 2π. Hence

$$m_n \geqq \frac{1}{1 - k_n^2}\left(\left|(1 - k_n^2) - (1 - \rho i e^{i\theta})^2\right| - k_n\left|1 - \rho i e^{i\theta}\right|^2\right).$$

An elementary argument shows that the minimum of the expression on the right is attained for $\theta = \pi/2$. Thus

$$m_n = \frac{\rho^2 + 2\rho - k_n}{1 + k_n}.$$

With the help of (7) we obtain the recurrence relation

$$m_n = \frac{\rho(2+\rho)m_{n-1} - \rho^2}{m_{n-1} + \rho^2}.$$

The proof of the lemma is now easily completed by induction.

LEMMA 2. *If the elements c_n satisfy condition* (3) *then*

$$1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 > \frac{1+\rho}{n+1+r},$$

where

$$r = \rho + 2\rho/(1-\rho).$$

It follows from Lemma 1 that

$$1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 \geqq 1 - \frac{\rho^2}{\rho^2(1 + (1-\rho^2)/(n - \rho(n-2)))^2}$$

$$= 1 - \left(\frac{n - \rho(n-2)}{n - \rho(n-2) + 1 - \rho^2} \right)^2$$

$$= \left(1 + \frac{n - \rho(n-2)}{n - \rho(n-2) + 1 - \rho^2} \right) \left(\frac{1-\rho^2}{n(1-\rho) + 2\rho + 1 - \rho^2} \right)$$

$$> \frac{1-\rho^2}{n(1-\rho) + 2\rho + 1 - \rho^2} = \frac{1+\rho}{n+1+r}.$$

LEMMA 3. *If the elements c_n satisfy condition* (3) *and if $c_{2n} \neq 0$, then*

$$\left| \frac{B_{2n}}{c_{2n} B_{2n-2}} \right| = \left| \frac{1}{c_{2n}} + c_{2n} + \theta_n \frac{c_{2n-1}^2}{c_{2n}} \right| \geqq \rho(1 + 1/(n+r)),$$

$n = 1, 2, \cdots,$ *provided $B_{-1} = 0$, $B_0 = 1$. Here $r = \rho + 2\rho/(1-\rho)$, and $\theta_n = B_{2n-3}/B_{2n-2}$.*

Applying the recurrence relation (4) twice to B_{2n} we obtain

$$\frac{B_{2n}}{B_{2n-2}} = 1 + c_{2n}^2 + c_{2n-1}^2 \cdot B_{2n-3}/B_{2n-2}.$$

Thus if we introduce θ_n and divide by c_{2n} we have

$$\frac{B_{2n}}{c_{2n} B_{2n-2}} = \frac{1}{c_{2n}} + c_{2n} + \theta_n c_{2n-1}^2 / c_{2n}.$$

Note that, in view of relation (5), $|\theta_n| \leqq 1/\rho$.

We now observe that the minimum of the above expression is obtained when the quantities c_n and θ_n lie on the boundaries of the regions over which they are permitted to vary. Hence

$$\left|1/c_{2n} + c_{2n} + \theta_n c_{2n-1}^2/c_{2n}\right| \geqq \left|1/c_{2n} + c_{2n}\right| - \left|\theta_n c_{2n-1}^2/c_{2n}\right|$$
$$\geqq \rho\left|\frac{2i + \rho e^{i\theta}}{i + \rho e^{i\theta}}\right| - \frac{\rho^2}{m_n\left|i + \rho e^{i\theta}\right|}.$$

The minimum value is obtained for $\theta=\pi/2$. This leads to

$$\left|1/c_{2n} + c_{2n} + \theta_n c_{2n-1}^2/c_{2n}\right| \geqq \rho\,\frac{(2+\rho)}{(1+\rho)} - \frac{\rho^2}{m_{n-1}(1+\rho)}$$

which, using the result of Lemma 1, becomes

$$\rho(1 + 1/(n + \rho + 2\rho/(1-\rho))).$$

We are now ready to prove

THEOREM 1. *The continued fraction* (1) *converges if for all* $n \geqq 1$

$$\left|c_{2n-1}\right| \leqq \rho, \qquad \left|c_{2n} \pm i\right| \geqq \rho,$$

where $0<\rho<1$.

To prove this theorem it suffices, as indicated in the beginning of this paper, to show that $\lim R_{2n}=0$. It is easily verified that

$$T_{2n}(z) = \frac{A_{2n-1}Z + c_{2n}^2 A_{2n-2}}{B_{2n-1}Z + c_{2n}^2 B_{2n-2}}.$$

From the theory of linear fractional transformations it follows that C_{2n}, the center of K_{2n}, is $T_{2n}(z^*)$, where z^* is the conjugate point with respect to the circle $|z-1|=\rho$ of the point z', which is defined by $T_{2n}(z')=\infty$. Now

$$z' = -\frac{c_{2n}^2 B_{2n-2}}{B_{2n-1}}.$$

Therefore

$$z^* = \frac{\overline{B}_{2n} - \rho^2\overline{B}_{2n-1}}{\overline{B}_{2n}},$$

and hence

$$C_{2n} = \frac{A_{2n}\overline{B}_{2n} - \rho^2 A_{2n-1}\overline{B}_{2n-1}}{\left|B_{2n}\right|^2 - \rho^2\left|B_{2n-1}\right|^2}.$$

Since $T_{2n}(1-\rho)$ is a point on the boundary of K_{2n} we can write

$$R_{2n} = |C_{2n} - T_{2n}(1-\rho)|$$
$$= \left| \frac{A_{2n}\overline{B}_{2n} - \rho^2 A_{2n-1}\overline{B}_{2n-1}}{|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2} - \frac{A_{2n} - \rho A_{2n-1}}{B_{2n} - \rho B_{2n-1}} \right|$$
$$= \left| \frac{\rho(\overline{B}_{2n} - \rho \overline{B}_{2n-1})(A_{2n}B_{2n-1} - A_{2n-1}B_{2n})}{(B_{2n} - \rho B_{2n-1})(|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2)} \right|$$
$$= \rho \prod_{v=1}^{2n} |c_v|^2 / |B_{2n}|^2 - \rho^2 |B_{2n-1}|^2$$
$$= \frac{\rho \prod_{v=1}^{n} |c_{2v-1}|^2}{1 - \rho^2 |B_{2n-1}/B_{2n}|^2 \prod_{v=1}^{n} |B_{2v}/c_{2v}B_{2v-2}|^2},$$

provided no $c_{2v} = 0$. If $c_{2m} = 0$, then $R_{2n} = 0$ for all $n > m$, and the theorem is proved. In the case where $c_{2v} \neq 0$ for all $v \geqq 1$ we apply Lemmas 1 and 2 and the fact that $|c_{2v-1}| < \rho$ to obtain

$$R_{2n} < \frac{n+1+r}{1+\rho}\rho^{2n+1} \Big/ \rho^{2n} \left(\prod_{v=1}^{n} \left(\frac{v+r+1}{v+r} \right) \right)^2$$
$$= \frac{\rho(n+1+r)}{(1+\rho)((n+1+r)/(1+r))^2} = \frac{\rho(1+r)^2}{(1+\rho)(n+1+r)}.$$

From this the theorem follows immediately.

We conclude with the following result.

THEOREM 2. *The continued fraction* (1) *converges at least in the wider sense if for all* $n \geqq 1$, $|c_{2n-1} \pm i| \geqq \rho$, $|c_{2n}| \leqq \rho$, *where* $0 < \rho < 1$.

For the proof of this theorem it suffices to point out that

$$1 + \frac{c_2^2}{1+} \cdots$$

converges by Theorem 1.

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WASHINGTON UNIVERSITY,
CARNEGIE INSTITUTE OF TECHNOLOGY, AND
UNIVERSITY OF COLORADO