A FAMILY OF BEST TWIN CONVERGENCE REGIONS
FOR CONTINUED FRACTIONS

VIKRAMADITYA SINGH AND W. J. THRON

Two regions \( E_1 \) and \( E_2 \) in the complex plane are called twin convergence regions for the continued fraction

\[
1 + \frac{c_1}{1 + \frac{c_2}{1 + \cdots}},
\]

if the fact that \( c_{2n-1} \subseteq E_1, c_{2n} \subseteq E_2, \) for all \( n \geq 1, \) is sufficient to insure the convergence of \((1)\). Twin convergence regions \( E_1 \) and \( E_2 \) are called best twin convergence regions, if there do not exist regions \( E'_1 \supseteq E_1 \) and \( E'_2 \supseteq E_2 \), where at least one of the equalities \( E'_1 = E_1, E'_2 = E_2 \) fails to hold, which are twin convergence regions for the continued fraction \((1)\).

Some time ago one of the authors \([1]\) proved that the continued fraction \((1)\) converges if

\[
|c_{2n-1}| \leq \rho, \quad |c_{2n} \pm i| \geq \rho + \epsilon, \quad n \geq 1,
\]

where \( 0 < \rho < 1 \) and \( \epsilon \) is an arbitrary positive quantity. It was also shown in \([1]\) that if the \( \epsilon \) could be removed in \((2)\) a set of best twin convergence regions would be obtained. It is the purpose of this paper to show that the \( \epsilon \) can indeed be removed.

Denote by \( t_n(z) \) the linear fractional transformation

\[
1 + \frac{c_n}{z}.
\]

It was shown in \([1]\) that if

\[
|c_{2n-1}| \leq \rho, \quad |c_{2n} \pm i| \geq \rho \quad \text{for all} \quad n \geq 1,
\]

then

\[
|1 - t_{2n-1}(z)| \leq \rho \quad \text{for all} \quad |z| \geq \rho,
\]

and

\[
|t_{2n}(z)| \geq \rho \quad \text{for all} \quad |z - 1| \leq \rho.
\]

Set

Presented to the Society, August 31, 1954; received by the editors July 6, 1954 and, in revised form, May 9, 1955.

1 This research was supported by the United States Air Force, through the Office of Scientific Research of the Air Research and Development Command.
\[ T_n(z) = t_1(t_2(\cdots t_n(z)\cdots)), \]

and denote by \( Z_1 \) and \( Z_2 \) the sets defined by \( |z| \geq \rho, |z-1| \leq \rho \), respectively. The sets

\[ K_{2n} = T_{2n}(Z_2), \quad K_{2n-1} = T_{2n-1}(Z_1) \]

are circular regions and \( |K_{n-1}| \leq \rho \) for all \( n \geq 2 \). Thus we have the well known (see for example [2, pp. 70–79]) "nest of circles." Finally, since \( 1 \in Z_1, 1 \in Z_2 \) and since the \( n \)th approximant of the continued fraction (1) is

\[ \frac{A_n}{B_n} = T_n(1), \]

we have for all \( n \geq m, A_n/B_n \in K_m \).

Let \( R_n \) be the radius of \( K_n \). Then \( \{R_n\} \) forms a nonincreasing sequence. Hence if it can be shown that a subsequence of \( \{R_n\} \) converges to zero it will follow that the continued fraction converges. Our proof consists in showing that \( \lim R_{2n} = 0 \).

We next observe that a consequence of the fundamental recurrence relation

\[ B_n = B_{n-1} + c_n B_{n-2}, \]

is that

\[ \frac{B_n}{B_{n-1}} = 1 + \frac{c_n}{1 + \frac{c_{n-1}}{1 + \frac{c_2}{1 + \cdots}}} \]

Thus

\[ \frac{B_n}{B_{n-1}} = t_1(\cdots t_2(1)\cdots). \]

From this it follows that

\[ \left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right| \leq \rho, \quad \left| \frac{B_{2n}}{B_{2n-1}} \right| \leq \rho. \]

The inequality (5) allows us to conclude that, if condition (3) holds, no \( B_n = 0 \). Now set

\[ m_n = \min \left| \frac{B_{2n}}{B_{2n-1}} \right|, \quad k_n = \max \left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right|. \]

Condition (5) insures that, for all \( n, m_n \geq \rho \) and \( k_n \leq \rho \). It is, however,
desirable to obtain sharper estimates, depending on \( n \), for these quantities. This is done in the following:

**Lemma 1.** If the elements \( c_n \) satisfy condition (3) then

\[
m_n = \rho (1 + (1 - \rho^2)/(n - \rho(n - 2))).
\]

Since

\[
\left| \frac{B_{2n-1}}{B_{2n-2}} - 1 \right| = \left| \frac{c_{2n-1}}{B_{2n-2}/B_{2n-3}} \right|,
\]

we have

\[
k_n = \rho^2/m_{n-1}.
\]

Equation (6) is readily verified for \( n = 1 \) since

\[
m_1 = \min |1 + c_2| = \rho(2 - \rho).
\]

Now

\[
m_n = \min \left| 1 + \frac{c_{2n}}{B_{2n-1}/B_{2n-3}} \right|,
\]

and one easily sees that the minimum is attained when \( c_{2n} \) is of the form \( i + pe^{i\theta} \) (we can assume that all \( c_n \) are in the upper half-plane) and \( B_{2n-1}/B_{2n-3} \) is of the form \( 1 + k_n e^{i\phi} \). Thus

\[
m_n = \min \left| 1 + \frac{(i + pe^{i\theta})^2}{1 + k_n e^{i\phi}} \right|, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq 2\pi.
\]

Now

\[
\left| 1 + \frac{(i + pe^{i\theta})^2}{1 + k_n e^{i\phi}} \right| = \left| 1 - \frac{(1 - \rho e^{i\phi})^2}{1 - k_n^2} (1 + k_n e^{i\phi}) \right|,
\]

where \( \phi' \) depends on \( k_n, \rho, \) and \( \phi \), but ranges from 0 to \( 2\pi \). Hence

\[
m_n \geq \frac{1}{1 - k_n^2} \left( |(1 - k_n^2) - (1 - \rho e^{i\phi})^2| - k_n |1 - \rho e^{i\phi}| \right).
\]

An elementary argument shows that the minimum of the expression on the right is attained for \( \theta = \pi/2 \). Thus

\[
m_n = \frac{\rho^2 + 2\rho - k_n}{1 + k_n}.
\]

With the help of (7) we obtain the recurrence relation
\[ m_n = \frac{\rho (2 + \rho) m_{n-1} - \rho^2}{m_{n-1} + \rho^2}. \]

The proof of the lemma is now easily completed by induction.

**Lemma 2.** If the elements \( c_n \) satisfy condition (3) then

\[ 1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 > \frac{1 + \rho}{n + 1 + r}, \]

where

\[ r = \rho + 2\rho/(1 - \rho). \]

It follows from Lemma 1 that

\[ 1 - \rho^2 \left| \frac{B_{2n-1}}{B_{2n}} \right|^2 \geq 1 - \frac{\rho^2}{\rho^2(1 + (1 - \rho^2)/(n - \rho(n - 2)))^2} \]

\[ = 1 - \left( \frac{n - \rho(n - 2)}{n - \rho(n - 2) + 1 - \rho^2} \right)^2 \]

\[ = \left( 1 + \frac{n - \rho(n - 2)}{n - \rho(n - 2) + 1 - \rho^2} \right) \left( \frac{1 - \rho^2}{n(1 - \rho) + 2\rho + 1 - \rho^2} \right) \]

\[ > \frac{1 - \rho^2}{n(1 - \rho) + 2\rho + 1 - \rho^2} = \frac{1 + \rho}{n + 1 + r}. \]

**Lemma 3.** If the elements \( c_n \) satisfy condition (3) and if \( c_{2n} \neq 0 \), then

\[ \left| \frac{B_{2n}}{c_{2n} B_{2n-2}} \right| = \left| \frac{1}{c_{2n}} + c_{2n} + \theta_n \frac{c_{2n-1}}{c_{2n}} \right| \geq \rho(1 + 1/(n + r)), \]

\( n = 1, 2, \cdots \), provided \( B_{-1} = 0, B_0 = 1 \). Here \( r = \rho + 2\rho/(1 - \rho) \), and \( \theta_n = B_{2n-4}/B_{2n-2} \).

Applying the recurrence relation (4) twice to \( B_{2n} \) we obtain

\[ \frac{B_{2n}}{B_{2n-2}} = 1 + \frac{2}{c_{2n}} + c_{2n-1} B_{2n-2}/B_{2n-2}. \]

Thus if we introduce \( \theta_n \) and divide by \( c_{2n} \) we have

\[ \frac{B_{2n}}{c_{2n} B_{2n-2}} = \frac{1}{c_{2n}} + c_{2n} + \theta_n c_{2n-1}/c_{2n}. \]

Note that, in view of relation (5), \( |\theta_n| \leq 1/\rho \).

We now observe that the minimum of the above expression is ob-
tained when the quantities \(c_n\) and \(\theta_n\) lie on the boundaries of the regions over which they are permitted to vary. Hence

\[
\frac{1}{c_{2n} + c_{2n} + \theta_n c_{2n-1}/c_{2n}} \geq \frac{1}{c_{2n} + c_{2n} - \theta_n c_{2n-1}/c_{2n}} \geq \rho \frac{2i + \rho e^{i\theta}}{i + \rho e^{i\theta}} - \frac{\rho^2}{m_n |i + \rho e^{i\theta}|}.
\]

The minimum value is obtained for \(\theta = \pi/2\). This leads to

\[
\frac{1}{c_{2n} + c_{2n} + \theta_n c_{2n-1}/c_{2n}} \geq \rho \frac{(2 + \rho)}{(1 + \rho)} - \frac{\rho^2}{m_{n-1}(1 + \rho)}
\]

which, using the result of Lemma 1, becomes

\[
\rho(1 + 1/(n + \rho + 2\rho/(1 - \rho))).
\]

We are now ready to prove

**Theorem 1.** The continued fraction (1) converges if for all \(n \geq 1\)

\[
| c_{2n-1} | \leq \rho, \quad | c_{2n} \pm i | \geq \rho,
\]

where \(0 < \rho < 1\).

To prove this theorem it suffices, as indicated in the beginning of this paper, to show that \(R_{2n} = 0\). It is easily verified that

\[
T_{2n}(z) = \frac{A_{2n-1}z + c_{2n}^2 A_{2n-2}}{B_{2n-1}z + c_{2n}^2 B_{2n-2}}.
\]

From the theory of linear fractional transformations it follows that \(C_{2n}\), the center of \(K_{2n}\), is \(T_{2n}(z^*)\), where \(z^*\) is the conjugate point with respect to the circle \(|z - 1| = \rho\) of the point \(z'\), which is defined by \(T_{2n}(z') = \infty\). Now

\[
z' = -\frac{c_{2n}^2 B_{2n-2}}{B_{2n-1}}.
\]

Therefore

\[
z^* = \frac{B_{2n} - \rho^2 B_{2n-1}}{B_{2n}},
\]

and hence

\[
C_{2n} = \frac{A_{2n} B_{2n} - \rho^2 A_{2n-1} B_{2n-1}}{|B_{2n}|^2 - \rho^2 |B_{2n-1}|^2}.
\]

Since \(T_{2n}(1 - \rho)\) is a point on the boundary of \(K_{2n}\) we can write
\[
R_{2n} = \left| C_{2n} - T_{2n}(1 - \rho) \right|
\]
\[
= \left| \frac{A_{2n}B_{2n} - \rho^2 A_{2n-1}B_{2n-1}}{B_{2n}} - \frac{A_{2n} - \rho A_{2n-1}}{B_{2n} - \rho B_{2n-1}} \right|
\]
\[
= \frac{\rho(B_{2n} - \rho B_{2n-1})(A_{2n}B_{2n-1} - A_{2n-1}B_{2n})}{(B_{2n} - \rho B_{2n-1})(B_{2n} - \rho^2 B_{2n-1})}
\]
\[
= \rho \prod_{v=1}^{2n} \left| c_v \right|^2 \left| B_{2n} \right|^2 - \rho^2 \left| B_{2n-1} \right|^2
\]
\[
= \frac{\rho \prod_{v=1}^{n} \left| c_{2v-1} \right|^2}{1 - \rho^2 \left| B_{2n-1}/B_{2n} \right|^2 \prod_{v=1}^{n} \left| B_{2v}/c_{2v}B_{2v-1} \right|^2}
\]
provided no \( c_{2v} = 0 \). If \( c_{2v} = 0 \), then \( R_{2n} = 0 \) for all \( n > m \), and the theorem is proved. In the case where \( c_{2v} \neq 0 \) for all \( v \geq 1 \) we apply Lemmas 1 and 2 and the fact that \( |c_{2v-1}| < \rho \) to obtain
\[
R_{2n} < \frac{n + 1 + r}{1 + \rho} \rho^{n+1} \left( \frac{n}{v + r} \right)^2 \cdot\frac{\rho(n + 1 + r)}{(1 + \rho)(n + 1 + r)/(1 + r)^2}
\]
From this the theorem follows immediately.

We conclude with the following result.

**Theorem 2.** The continued fraction (1) converges at least in the wider sense if for all \( n \geq 1 \), \( |c_{2n-1} \pm i| \geq \rho \), \( |c_{2n}| \leq \rho \), where \( 0 < \rho < 1 \).

For the proof of this theorem it suffices to point out that
\[
1 + \frac{c_2}{1 + \cdots}
\]
converges by Theorem 1.

**References**