ON THE LIMIT OF THE COEFFICIENTS OF THE EIGENFUNCTION SERIES ASSOCIATED WITH A CERTAIN NON-SELF-ADJOINT DIFFERENTIAL SYSTEM

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Introduction. In the attempt to solve certain problems in mathematical-physics, such as diffraction of an arbitrary pulse by a wedge as considered by Irvin Kay [1], one encounters the hyperbolic differential equation

\[ u_{xx} - q(x)u = u_{zt} - p(x)u_t \]

where \( u(x, t) \) must satisfy the conditions \( u(a, t) = u(b, t) = 0 \) and \( u(x, 0) = F(x) \). In attempting to solve equation (1) by separation of variables, one is led to the consideration of expanding an arbitrary function \( F(x) \) in terms of the eigenfunctions, or nonzero solutions, \( u_n(x) \) of the equation:

\[ (A + \lambda B)u = 0 \]

satisfying the conditions \( u(a) = u(b) = 0 \), where \( A \) is the operator \( d^2/dx^2 + q(x) \) and \( B \) is the operator \(-d/dx + p(x)\). The system adjoint to (2) is:

\[ (A^* + \lambda B^*)v = 0, \quad v(a) = v(b) = 0 \]

where \( A = A^* \) and \( B^* = d/dx + p(x) \).

Conditions have been established [2], under which a function \( F(x) \) of bounded variation on \((a, b)\) can be expanded in terms of \( u_n(x) \). However, in the expansion \( F(x) = \sum_{n=1}^{\infty} a_n u_n(x) \) there are certain properties of the coefficients, \( a_n \), which differ quite radically from the corresponding properties of the coefficients of certain well-known self-adjoint eigenfunction expansions. For example, if \( B_n \) are the Fourier coefficients of a function \( g(x) \), it is well known that \( \lim_{n \to \infty} B_n = 0 \). However, in the expansion \( F(x) = \sum_{n=1}^{\infty} a_n u_n(x) \), it is found that \( \lim_{n \to \infty} a_n \) is not in general equal to zero. Consequently, the series \( \sum_{n=1}^{\infty} a_n^2 \), unlike the corresponding series of Fourier coefficients, does not in general converge.

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In this paper it is proved that \( \lim_{n \to \infty} a_n = 0 \) if and only if \( F(a) = F(b) = 0 \).

The following theorem was proved in [2]:

**Theorem 1.** Let \( g(x) \) be continuous and let \( p(x) \) have a continuous second derivative. If \( F(x) \) is of bounded variation in \((a, b)\) and if

\[
F(a + 0) + F(b - 0) \exp \left[ - \int_a^b p(t) dt \right] = 0,
\]

then the series

\[
\sum_{n=0}^{\infty} a_n u_n(x),
\]

where

\[
a_n = \frac{\int_a^b F(\xi) B^* v_n(\xi) d\xi}{\int_a^b u_n(\xi) B^* v_n(\xi) d\xi}
\]

with \( u_n(x) \) and \( v_n(x) \) eigenfunctions of (2) and (3) respectively, converges to \( [F(x+0) + F(x-0)]/2 \) in the interval \( a < x < b \). If \( F(x) \) does not satisfy the condition (4), then the series (5) converges to

\[
T(x) = [F(x + 0) + F(x - 0)]/2 - c \exp \left[ \int_a^x p(t) dt \right]
\]

in the interval \( a < x < b \), where

\[
c = \left\{ F(a + 0) + F(b - 0) \exp \left[ - \int_a^b p(t) dt \right] \right\} / 2.
\]

We now prove:

**Theorem 2.** If \( F'(x) \) exists and is of bounded variation for \( a \leq x \leq b \), and if \( F(a) + F(b) \exp \left[ - \int_a^b p(t) dt \right] = 0 \), then a necessary and sufficient condition that \( \lim_{n \to \infty} a_n = 0 \) is that \( F(a) = F(b) = 0 \). (The prime denotes differentiation with respect to \( x \).)

**Asymptotic form of \( a_n \).** Since

\[
a_n = \frac{\int_a^b F(\xi) B^* v_n(\xi) d\xi}{\int_a^b u_n(\xi) B^* v_n(\xi) d\xi},
\]

we can develop the asymptotic form for \( a_n \) by considering the corresponding forms for \( u_n(\xi) \) and \( B^* v_n(\xi) \), and we have from [2]
\[ u_n(x) = u_m(x, \lambda_n) \]
\[ = \lambda_n^{-1} \left\{ \exp \left[ \lambda_n(x - a) - \int_a^x \phi(t) dt \right] - \exp \left[ \int_a^x \phi(t) dt \right] \right\} \]
\[ + O(\lambda_n^{-2} \exp \left[ (\lambda_n + | \sigma |)(x - a)/2 \right]), \]
where \( \sigma_n = \Re \lambda_n \) and \( | \lambda_n | \to \infty \),
\[ B^* v_n(x) = B^* v_m(x, \lambda_n) = \exp \left[ -\lambda_n x + \int_a^x \phi(t) dt \right] + \Omega_n, \]
where
\[ \Omega_n = \begin{cases} O(\lambda_n^{-2} \exp [-\lambda_n a]) + O(\lambda_n^{-1} \exp [-\lambda_n x]), & \Re \lambda_n \geq 0, \\ O(\lambda_n^{-1} \exp [-\lambda_n x]), & \Re \lambda_n \leq 0, \end{cases} \]
and where
\[ 2n \pi i + 2 \int_a^b \phi(t) dt + O \left( \frac{1}{n} \right) \]
\[ \lambda_n = \frac{2n \pi i}{b - a} \cdot \]

From (6) and (7) we have:
\[ u_n(\xi) B^* v_n(\xi) = \frac{\exp (-\lambda_n \xi)}{\lambda_n} + O \left( \frac{1}{\lambda_n^2} \right) \]
\[ \exp \left[ -\lambda_n \xi + 2 \int_a^\xi \phi(t) dt \right] \]
and
\[ \int_a^b u_n(\xi) B^* v_n(\xi) d\xi = \int_a^b \frac{\exp (-\lambda_n \xi)}{\lambda_n} d\xi + \int_a^b O \left( \frac{1}{\lambda_n^2} \right) d\xi \]
\[ - \int_a^b \frac{\exp \left[ -\lambda_n \xi + 2 \int_a^\xi \phi(t) dt \right]}{\lambda_n} d\xi. \]

Now since \( u_n(\xi) \) has a bounded derivative on \((a, b)\), it follows that \( u_n(\xi) \) is of bounded variation for \( a \leq \xi \leq b \). Also
\[ B^* v_n(\xi) = \rho(\xi) v_n(\xi) + v_n'(\xi) \]
and
\[
\frac{d}{d\xi} \left[ B^* v_n(\xi) \right] = p(\xi)v_n'(\xi) + v_n(\xi)p'(\xi) + v_n''(\xi)
\]

but by (3)

\[
v_n''(\xi) = -q(\xi)v_n(\xi) - \lambda_n[p(\xi)v_n(\xi) + v'_n(\xi)].
\]

Therefore

\[
\frac{d}{d\xi} \left[ B^* v_n(\xi) \right] = p(\xi)v_n'(\xi) + v_n(\xi)p'(\xi) - q(\xi)v_n(\xi)
- \lambda_n[p(\xi)v_n(\xi) + v'_n(\xi)]
\]

is bounded for \(a \leq \xi \leq b\). Hence \(B^* v_n(\xi)\) is of bounded variation for \(a \leq \xi \leq b\) and it follows that

\[
\mu_n(\xi) B^* v_n(\xi)
\]

is of bounded variation, and consequently the term

\[
O\left(\frac{1}{\lambda_n^2}\right) = \frac{g(n, \xi)}{\lambda_n^2}
\]

in (9) is of bounded variation for \(a \leq \xi \leq b\). Now put

\[
g(n, \xi) = Q_1(n, \xi) - Q_2(n, \xi)
\]

where \(Q_1(n, \xi)\) and \(Q_2(n, \xi)\) are two non-negative bounded monotone decreasing functions, and apply the mean value theorem to (9) and we get:

\[
\int_a^b \mu_n(\xi) B^* v_n(\xi) d\xi
= \int_a^b \exp \left( -\lambda_n \xi \right) d\xi + \frac{Q_1(n, a)}{\lambda_n^2} \int_a^b d\xi
- \frac{Q_2(n, a)}{\lambda_n^2} \int_a^b \frac{\exp \left( -\lambda_n \xi + 2 \int_a^\xi p(t) dt \right)}{\lambda_n} d\xi
\]

\[
= \frac{(b - a) \exp \left( -\lambda_n a \right)}{\lambda_n} + O\left(\frac{1}{\lambda_n^3}\right)
= \frac{\left[ (b - a) \exp \left( -\lambda_n a \right) + O\left(\frac{1}{\lambda_n}\right) \right]}{\lambda_n}, \quad \text{where} \quad a < d_1, d_2 < b.
\]
Now put

\[ (b - a) \exp (-\lambda_n a) + O\left(\frac{1}{\lambda_n}\right) = \frac{1}{B(n)}. \]

We then have

\[ a_n = B(n)\lambda_n \int_a^b F(\xi) \exp \left[ -\lambda_n \xi + \int_a^\xi \phi(t) dt \right] d\xi \]

\[ + B(n)\lambda_n \int_a^b \frac{F(\xi)O\left[\exp \left(-\lambda_n \xi\right)\right]}{\lambda_n} d\xi, \]

\[ + B(n)\lambda_n \int_a^b \frac{F(\xi)O\left[\frac{\exp \left(-\lambda_n \xi\right)}{\lambda_n^2}\right]}{\lambda_n} d\xi, \quad \text{for Re } \lambda_n \geq 0, \]

\[ a_n = B(n)\lambda_n \int_a^b F(\xi) \exp \left[ -\lambda_n \xi + \int_a^\xi \phi(t) dt \right] d\xi \]

\[ + B(n)\lambda_n \int_a^b \frac{F(\xi)O\left[\left(-\lambda_n \xi\right)\right]}{\lambda_n} d\xi, \quad \text{Re } \lambda_n \leq 0. \]

**Determination of the limit of** \( a_n \). From equation (14) we have, for \( \text{Re } \lambda_n \geq 0 \):

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} B(n)\lambda_n \int_a^b F(\xi) \exp \left[ -\lambda_n \xi + \int_a^\xi \phi(t) dt \right] d\xi \]

\[ + \lim_{n \to \infty} B(n) \int_a^b F(\xi)O\left[\exp \left(-\lambda_n \xi\right)\right] d\xi \]

\[ + \lim_{n \to \infty} \frac{B(n)}{\lambda_n} \int_a^b F(\xi)O\left[\left(-\lambda_n \xi\right)\right] d\xi, \]

provided these limits exist. Since \( B^* u_n(x) \) is of bounded variation for \( a \leq x \leq b \), it is clear from (7) that the expressions \( O\exp \left[-\lambda_n \xi\right] \) and \( O\exp \left[-\lambda_n a\right] \) in the integrands of (16) are also of bounded variation for \( a \leq \xi \leq b \).

Consider now the second integral of (16). We have:

\[ \lim_{n \to \infty} B(n) \int_a^b F(\xi)O\left[\left(-\lambda_n \xi\right)\right] d\xi \]

\[ = \lim_{n \to \infty} B(n) \int_a^b F(\xi)g_1(n, \xi) \exp \left[-\lambda_n \xi\right] d\xi, \]

where \( g_1(n, \xi) \) and consequently \( F(\xi)g_1(n, \xi) \) are of bounded variation.
for \( a \leq \xi \leq b \). Put \( F(\xi)g_1(n, \xi) = Q_3(n, \xi) - Q_4(n, \xi) \), where \( Q_3(n, \xi) \) and \( Q_4(n, \xi) \) are two non-negative, bounded monotone decreasing functions, and apply the mean value theorem to (17) and we get:

\[
\lim_{n \to \infty} B(n) \int_a^b F(\xi)g_1(n, \xi) \exp \left[-\lambda_n \xi\right] d\xi
\]

(18) \[
= \lim_{n \to \infty} B(n) \left[ \frac{Q_3(n, a) \exp \left[-\lambda_n a\right]}{-\lambda_n} a + \frac{Q_4(n, a) \exp \left[-\lambda_n a\right]}{\lambda_n} a \right]
\]

\[
= \lim_{n \to \infty} \frac{O(1)}{\lambda_n}.
\]

Since \( \lambda_n = \frac{(2n\pi i + 2 \int_a^b \rho(t) dt)/(b-a) + O(1/n)}{n} \), it is clear that \( \lambda_n = nO(1) \) and (18) becomes:

(19) \[
\lim_{n \to \infty} B(n) \int_a^b F(\xi)g_1(n, \xi) \exp \left[-\lambda_n \xi\right] d\xi = \lim_{n \to \infty} \frac{O(1)}{n} = 0.
\]

A completely similar argument will show that the third integral in (16) becomes:

\[
\lim_{n \to \infty} \frac{B(n)}{\lambda_n} \int_a^b F(\xi)O(\exp \left[-\lambda_n a\right]) d\xi = \lim_{n \to \infty} \frac{O(1)}{n} = 0.
\]

Hence, by (14) and (15), we have:

(20) \[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \lambda_n B(n) \int_a^b F(\xi) \exp \left[-\lambda_n \xi + \int_a^\xi \rho(t) dt\right] d\xi.
\]

If in (20), we put \( F(\xi) \exp \left[\int_a^\xi \rho(t) dt\right] = H(\xi) \) and integrate by parts we get:

\[
\lim_{n \to \infty} a_n = - \lim_{n \to \infty} B(n) \left\{ \left[H(\xi) \exp (-\lambda_n \xi)\right]_a^b \right. \\
- \left. \int_a^b H'(\xi) \exp (-\lambda_n \xi) d\xi \right\}.
\]

(21)

Using the fact that \( F'(\xi) \) is of bounded variation for \( a \leq \xi \leq b \), it follows that \( H'(\xi) \) is also of bounded variation for \( a \leq \xi \leq b \) and

(22) \[
\lim_{n \to \infty} \int_a^b H'(\xi) \exp (-\lambda_n \xi) d\xi = \lim_{n \to \infty} O\left(\frac{1}{\lambda_n}\right) = \lim_{n \to \infty} \frac{O(1)}{n} = 0.
\]

We have finally, by (7a), (21) and (22):
\[
\lim_{n \to \infty} a_n = \lim_{n \to \infty} B(n) \left\{ F(a) \exp \left[ -\lambda_n a \right] - F(b) \exp \left[ \int_a^b p(t) dt - \lambda_n b \right] \right\}.
\]

But, by Theorem 2, we have

\[
F(a) = -\exp \left[ -\int_a^b p(t) dt \right] F(b).
\]

Therefore

\[
\lim_{n \to \infty} a_n = -F(b) \lim_{n \to \infty} B(n) \left( \exp \left[ -\int_a^b p(t) dt - \lambda_n a \right] + \exp \left[ \int_a^b p(t) dt - \lambda_n b \right] \right).
\]

We have from (13) that:

\[
\lim_{n \to \infty} B(n) e^{-\lambda_n a} = \frac{1}{b - a}.
\]

Using this result in (25), we have

\[
\lim_{n \to \infty} a_n = -\frac{F(b)}{b - a} \lim_{n \to \infty} \left( \exp \left[ -\int_a^b p(t) dt - 2\lambda_n a \right] + \exp \left[ \int_a^b p(t) dt - \lambda_n (b + a) \right] \right).
\]

From the value of \( \lambda_n \) as given by (7a), it is clear that the second factor in equation (26) does not approach zero as \( n \) approaches infinity. Hence it follows that \( \lim_{n \to \infty} a_n = 0 \) if and only if \( F(b) = 0 \). Then it follows from (24) that \( F(a) \) is also equal to zero. And, our theorem is proved.

References


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