ON THE LIMIT OF THE COEFFICIENTS OF THE EIGENFUNCTION SERIES ASSOCIATED WITH A CERTAIN NON-SELF-ADJOINT DIFFERENTIAL SYSTEM[1]

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Introduction. In the attempt to solve certain problems in mathematical-physics, such as diffraction of an arbitrary pulse by a wedge as considered by Irvin Kay [1], one encounters the hyperbolic differential equation

$$u_{xx} - q(x)u = u_{xt} - p(x)u_t \tag{1}$$

where $u(x, t)$ must satisfy the conditions $u(a, t) = u(b, t) = 0$ and $u(x, 0) = F(x)$. In attempting to solve equation (1) by separation of variables, one is led to the consideration of expanding an arbitrary function $F(x)$ in terms of the eigenfunctions, or nonzero solutions, $u_n(x)$ of the equation:

$$(A + \lambda B)u = 0 \tag{2}$$

satisfying the conditions $u(a) = u(b) = 0$, where A is the operator $d^2/dx^2 + q(x)$ and B is the operator $-d/dx + p(x)$. The system adjoint to (2) is:

$$(A^* + \lambda B^*)v = 0, \qquad v(a) = v(b) = 0 \tag{3}$$

where $A = A^*$ and $B^* = d/dx + p(x)$.

Conditions have been established [2], under which a function $F(x)$ of bounded variation on (a, b) can be expanded in terms of $u_n(x)$. However, in the expansion $F(x) = \sum_{-\infty}^{\infty} a_n u_n(x)$ there are certain properties of the coefficients, a_n, which differ quite radically from the corresponding properties of the coefficients of certain well-known self-adjoint eigenfunction expansions. For example, if B_n are the Fourier coefficients of a function $g(x)$, it is well known that $\lim_{n\to\infty} B_n = 0$. However, in the expansion $F(x) = \sum_{-\infty}^{\infty} a_n u_n(x)$, it is found that $\lim_{n\to\infty} a_n$ is not in general equal to zero. Consequently, the series $\sum_1^{\infty} a_n^2$, unlike the corresponding series of Fourier coefficients, does not in general converge.

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In this paper it is proved that $\lim_{n\to\infty} a_n=0$ if and only if $F(a)=F(b)=0$.

The following theorem was proved in [2]:

THEOREM 1. *Let $q(x)$ be continuous and let $p(x)$ have a continuous second derivative. If $F(x)$ is of bounded variation in (a, b) and if*

$$F(a+0)+F(b-0)\exp\left[-\int_a^b p(t)dt\right]=0, \tag{4}$$

then the series

$$\sum_{-\infty}^{\infty} a_n u_n(x), \tag{5}$$

where

$$a_n=\frac{\int_a^b F(\xi)B^*v_n(\xi)d\xi}{\int_a^b u_n(\xi)B^*v_n(\xi)d\xi}$$

with $u_n(x)$ and $v_n(x)$ eigenfunctions of (2) *and* (3) *respectively, converges to $[F(x+0)+F(x-0)]/2$ in the interval $a<x<b$. If $F(x)$ does not satisfy the condition* (4), *then the series* (5) *converges to*

$$T(x)=[F(x+0)+F(x-0)]/2-c\exp\left[\int_a^x p(t)dt\right]$$

in the interval $a<x<b$, where

$$c=\left\{F(a+0)+F(b-0)\exp\left[-\int_a^b p(t)dt\right]\right\}\Big/2.$$

We now prove:

THEOREM 2. *If $F'(x)$ exists and is of bounded variation for $a\leqq x\leqq b$, and if $F(a)+F(b)\exp\left[-\int_a^b p(t)dt\right]=0$, then a necessary and sufficient condition that $\lim_{n\to\infty} a_n=0$ is that $F(a)=F(b)=0$.* (*The prime denotes differentiation with respect to x.*)

Asymptotic form of a_n. Since

$$a_n=\int_a^b F(\xi)B^*v_n(\xi)d\xi\Big/\int_a^b u_n(\xi)B^*v_n(\xi)d\xi,$$

we can develop the asymptotic form for a_n by considering the corresponding forms for $u_n(\xi)$ and $B^*v_n(\xi)$, and we have from [2]

$$
\begin{aligned}
u_n(x) &= u_a(x, \lambda_n) \\
&= \lambda_n^{-1}\left\{\exp\left[\lambda_n(x-a) - \int_a^x p(t)dt\right] - \exp\left[\int_a^x p(t)dt\right]\right\} \qquad (6)\\
&\quad + O(\lambda_n^{-2}\exp\left[(\lambda_n + |\sigma|)(x-a)/2\right]),
\end{aligned}
$$

where $\sigma_n = \operatorname{Re}\lambda_n$ and $|\lambda_n| \to \infty$,

$$B^*v_n(x) = B^*v_a(x, \lambda_n) = \exp\left[-\lambda_n x + \int_a^x p(t)dt\right] + \Omega_a, \tag{7}$$

where

$$\Omega_a = \begin{cases} O(\lambda_n^{-2}\exp[-\lambda_n a]) + O(\lambda_n^{-1}\exp[-\lambda_n x]), & \operatorname{Re}\lambda_n \geqq 0, \\ O(\lambda_n^{-1}\exp[-\lambda_n x]) & \operatorname{Re}\lambda_n \leqq 0, \end{cases}$$

and where

$$\lambda_n = \frac{2n\pi i + 2\int_a^b p(t)dt + O\left(\frac{1}{n}\right)}{b-a}. \tag{7a}$$

From (6) and (7) we have:

$$u_n(\xi)B^*v_n(\xi) = \frac{\exp(-\lambda_n a)}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right) - \frac{\exp\left[-\lambda_n\xi + 2\int_a^\xi p(t)dt\right]}{\lambda_n} \tag{8}$$

and

$$\int_a^b u_n(\xi)B^*v_n(\xi)d\xi = \int_a^b \frac{\exp(-\lambda_n a)}{\lambda_n}d\xi + \int_a^b O\left(\frac{1}{\lambda_n^2}\right)d\xi - \int_a^b \frac{\exp\left[-\lambda_n\xi + 2\int_a^\xi p(t)dt\right]}{\lambda_n}d\xi. \tag{9}$$

Now since $u_n(\xi)$ has a bounded derivative on (a, b), it follows that $u_n(\xi)$ is of bounded variation for $a \leqq \xi \leqq b$. Also

$$B^*v_n(\xi) = p(\xi)v_n(\xi) + v_n'(\xi)$$

and

$$\frac{d}{d\xi}\left[B^*v_n(\xi)\right] = p(\xi)v_n'(\xi) + v_n(\xi)p'(\xi) + v_n''(\xi)$$

but by (3)

$$v_n''(\xi) = -q(\xi)v_n(\xi) - \lambda_n\left[p(\xi)v_n(\xi) + v_n'(\xi)\right].$$

Therefore

$$\frac{d}{d\xi}\left[B^*v_n(\xi)\right] = p(\xi)v_n'(\xi) + v_n(\xi)p'(\xi) - q(\xi)v_n(\xi) - \lambda_n\left[p(\xi)v_n(\xi) + v_n'(\xi)\right] \tag{10}$$

is bounded for $a \leqq \xi \leqq b$. Hence $B^*v_n(\xi)$ is of bounded variation for $a \leqq \xi \leqq b$ and it follows that

$$u_n(\xi)B^*v_n(\xi) \tag{11}$$

is of bounded variation, and consequently the term

$$O\left(\frac{1}{\lambda_n^2}\right) \equiv \frac{g(n, \xi)}{\lambda_n^2}$$

in (9) is of bounded variation for $a \leqq \xi \leqq b$. Now put

$$g(n, \xi) = Q_1(n, \xi) - Q_2(n, \xi)$$

where $Q_1(n, \xi)$ and $Q_2(n, \xi)$ are two non-negative bounded monotone decreasing functions, and apply the mean value theorem to (9) and we get:

$$\begin{aligned}
&\int_a^b u_n(\xi)B^*v_n(\xi)d\xi \\
&= \int_a^b \frac{\exp(-\lambda_n a)}{\lambda_n}\,d\xi + \frac{Q_1(n, a)}{\lambda_n^2}\int_a^{d_1} d\xi \\
&\quad - \frac{Q_2(n, a)}{\lambda_n^2}\int_a^{d_2} d\xi - \int_a^b \frac{\exp\left[-\lambda_n\xi + 2\int_a^\xi p(t)dt\right]}{\lambda_n}\,d\xi \\
&= \frac{(b-a)\exp(-\lambda_n a)}{\lambda_n} + O\left(\frac{1}{\lambda_n^2}\right) \\
&= \frac{\left[(b-a)\exp(-\lambda_n a) + O\left(\dfrac{1}{\lambda_n}\right)\right]}{\lambda_n}, \quad \text{where } a < d_1, d_2 < b.
\end{aligned} \tag{12}$$

Now put

$$\left[(b-a)\exp(-\lambda_n a)+O\left(\frac{1}{\lambda_n}\right)\right]=\frac{1}{B(n)}. \tag{13}$$

We then have

$$\begin{aligned} a_n &= B(n)\lambda_n\int_a^b F(\xi)\exp\left[-\lambda_n\xi+\int_a^\xi p(t)dt\right]d\xi \\ &\quad + B(n)\lambda_n\int_a^b \frac{F(\xi)O[\exp(-\lambda_n\xi)]}{\lambda_n}d\xi \\ &\quad + B(n)\lambda_n\int_a^b F(\xi)O\left(\frac{\exp[-\lambda_n a]}{\lambda_n^2}\right)d\xi, \qquad \text{for } \operatorname{Re}\lambda_n \geqq 0, \end{aligned} \tag{14}$$

$$\begin{aligned} a_n &= B(n)\lambda_n\int_a^b F(\xi)\exp\left[-\lambda_n\xi+\int_a^\xi p(t)dt\right]d\xi \\ &\quad + B(n)\lambda_n\int_a^b F(\xi)O\left(\frac{\exp[-\lambda_n\xi]}{\lambda_n}\right)d\xi, \qquad \operatorname{Re}\lambda_n \leqq 0. \end{aligned} \tag{15}$$

Determination of the limit of a_n. From equation (14) we have, for $\operatorname{Re}\lambda_n \geqq 0$:

$$\begin{aligned} \lim_{n\to\infty} a_n &= \lim_{n\to\infty} B(n)\lambda_n\int_a^b F(\xi)\exp\left[-\lambda_n\xi+\int_a^\xi p(t)dt\right]d\xi \\ &\quad + \lim_{n\to\infty} B(n)\int_a^b F(\xi)O(\exp[-\lambda_n\xi])d\xi \\ &\quad + \lim_{n\to\infty}\frac{B(n)}{\lambda_n}\int_a^b F(\xi)O(\exp[-\lambda_n a])d\xi, \end{aligned} \tag{16}$$

provided these limits exist. Since $B^*v_n(x)$ is of bounded variation for $a\leqq x\leqq b$, it is clear from (7) that the expressions $O(\exp[-\lambda_n\xi])$ and $O(\exp[-\lambda_n a])$ in the integrands of (16) are also of bounded variation for $a\leqq\xi\leqq b$.

Consider now the second integral of (16). We have:

$$\begin{aligned} &\lim_{n\to\infty} B(n)\int_a^b F(\xi)O(\exp[-\lambda_n\xi])d\xi \\ &\qquad = \lim_{n\to\infty} B(n)\int_a^b F(\xi)g_1(n,\xi)\exp[-\lambda_n\xi]d\xi, \end{aligned} \tag{17}$$

where $g_1(n,\xi)$ and consequently $F(\xi)g_1(n,\xi)$ are of bounded variation for $a \leqq \xi \leqq b$. Put $F(\xi)g_1(n, \xi) = Q_3(n, \xi) - Q_4(n, \xi)$, where $Q_3(n, \xi)$ and $Q_4(n, \xi)$ are two non-negative, bounded monotone decreasing functions, and apply the mean value theorem to (17) and we get:

$$\lim_{n\to\infty} B(n) \int_a^b F(\xi)g_1(n, \xi) \exp\,[-\lambda_n\xi]d\xi$$

$$(18) \quad = \lim_{n\to\infty} B(n)\left[\left.\frac{Q_3(n, a)\exp\,[-\lambda_n\xi]}{-\lambda_n}\right)_a^{d_3} + \left.\frac{Q_4(n, a)\exp\,[-\lambda_n\xi]}{\lambda_n}\right)_a^{d_4}\right]$$

$$= \lim_{n\to\infty} \frac{O(1)}{\lambda_n}\cdot$$

Since $\lambda_n = (2n\pi i + 2\int_a^b p(t)dt)/(b-a) + O(1/n)$, it is clear that $\lambda_n = nO(1)$ and (18) becomes:

$$(19) \qquad \lim_{n\to\infty} B(n) \int_a^b F(\xi)g_1(n, \xi) \exp\,[-\lambda_n\xi]d\xi = \lim_{n\to\infty} \frac{O(1)}{n} = 0.$$

A completely similar argument will show that the third integral in (16) becomes:

$$\lim_{n\to\infty} \frac{B(n)}{\lambda_n} \int_a^b F(\xi)O(\exp\,[-\lambda_n a])d\xi = \lim_{n\to\infty} \frac{O(1)}{n} = 0.$$

Hence, by (14) and (15), we have:

$$(20) \qquad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \lambda_n B(n) \int_a^b F(\xi) \exp\left[-\lambda_n\xi + \int_a^\xi p(t)dt\right]d\xi.$$

If in (20), we put $F(\xi)\exp\,[\int_a^\xi p(t)dt] = H(\xi)$ and integrate by parts we get:

$$(21) \qquad \begin{aligned} \lim_{n\to\infty} a_n = -\lim_{n\to\infty} B(n) \Big\{ &[H(\xi)\exp\,(-\lambda_n\xi)]_a^b \\ &- \int_a^b H'(\xi)\exp\,(-\lambda_n\xi)d\xi \Big\}. \end{aligned}$$

Using the fact that $F'(\xi)$ is of bounded variation for $a \leqq \xi \leqq b$, it follows that $H'(\xi)$ is also of bounded variation for $a \leqq \xi \leqq b$ and

$$(22) \quad \lim_{n\to\infty} \int_a^b H'(\xi)\exp\,(-\lambda_n\xi)d\xi = \lim_{n\to\infty} O\left(\frac{1}{\lambda_n}\right) = \lim_{n\to\infty} \frac{O(1)}{n} = 0.$$

We have finally, by (7a), (21) and (22):

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} B(n)\left\{F(a)\exp\left[-\lambda_n a\right] - F(b)\exp\left[\int_a^b p(t)dt - \lambda_n b\right]\right\}. \tag{23}$$

But, by Theorem 2, we have

$$F(a) = -\exp\left[-\int_a^b p(t)dt\right]F(b). \tag{24}$$

Therefore

$$\lim_{n\to\infty} a_n = -F(b)\lim_{n\to\infty} B(n)\left(\exp\left[-\int_a^b p(t)dt - \lambda_n a\right] + \exp\left[\int_a^b p(t)dt - \lambda_n b\right]\right). \tag{25}$$

We have from (13) that:

$$\lim_{n\to\infty} B(n)e^{-\lambda_n a} = \frac{1}{b-a}\cdot$$

Using this result in (25), we have

$$\lim_{n\to\infty} a_n = \frac{-F(b)}{b-a}\lim_{n\to\infty}\left(\exp\left[-\int_a^b p(t)dt - 2\lambda_n a\right] + \exp\left[\int_a^b p(t)dt - \lambda_n(b+a)\right]\right). \tag{26}$$

From the value of λ_n as given by (7a), it is clear that the second factor in equation (26) does not approach zero as n approaches infinity. Hence it follows that $\lim_{n\to\infty} a_n = 0$ if and only if $F(b) = 0$. Then it follows from (24) that $F(a)$ is also equal to zero. And, our theorem is proved.

References

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2. B. Friedman and L. I. Mishoe, *Eigenfunction expansions associated with a non-self-adjoint differential equation,* Institute of Mathematical Sciences, New York University, Research Report No. BR-4, 1954.

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