ON D. E. LITTLEWOOD'S ALGEBRA OF S-FUNCTION[1]

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The problem of finding the number of concomitants of a given degree and type is found to reduce to the essential problem of expressing the induced matrix of an induced matrix as the direct sum of irreducible invariant matrices.

According to Schur's definition of an invariant matrix an invariant matrix of an invariant matrix is clearly an invariant matrix though not in general irreducible.

Thus $[A^{\{\lambda\}}]^{\{\mu\}} = \dot{\sum} K_{\lambda\mu\nu} A^{\{\nu\}}$, the symbol $\dot{\sum}$ denoting direct sum and $A^{\{\lambda\}}$ denoting the invariant matrix of A corresponding to the partition $(\lambda) = (\lambda_1\lambda_2 \cdots \lambda_i)$.

Hence Littlewood was able to define a new multiplication of S-functions $\{\lambda\} \otimes \{\mu\} = \sum K_{\lambda\mu\nu}\{\nu\}$ where the operation denoted by $\otimes$ is called the plethysm of S-function and the expression $\{\lambda\} \otimes \{\mu\}$ is read "λ plethys μ."

One of the main problems of the invariant theory is to evaluate $\{\lambda\} \otimes \{\mu\}$ where (λ) may be a partition of (m) and (μ) a partition of n.

Littlewood was first to tackle successfully this problem by applying different methods [**7**; **8**]. One of his best is "the third method." He found that if $\{\lambda\} \otimes \{n\} = \Sigma\{\nu\}$ then $\sum g_{1\zeta\nu}\{\zeta\} = \{\lambda\} \otimes \{n-1\} \cdot [\sum g_{1\mu\lambda}\{\mu\}]$ where g_{rst} is defined from the multiplication of S-function by means of $\{r\}\{s\} = g_{rst}\{t\}$. For small degrees if $\sum g_{1\zeta\nu}(\zeta)$ is known then $\Sigma\{\nu\}$ may be easily inferred and this is the required expansion of $\{\lambda\} \otimes \{n\}$. As the degrees get larger certain alternatives present themselves and it becomes difficult to select the correct $\{\nu\}$. However, the method may still be valuable if used in conjunction with some other method which would indicate the correct choice when difficulty arises.

Later several attacks (Duncan [**1**; **2**], Foulkes [**3**; **4**], Murnaghan [**9**; **10**; **11**], Newell [**12**], Robinson [**13**; **14**], Todd [**16**], Thrall [**15**], Zia-Ud-Din [**17**]) have been made on this problem, varying considerably in their generality and the degree to which the results obtained have been applicable to numerical cases.

In this paper the use of a general theorem taken in conjunction with

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Littlewood's third method has been found to be quite enough to calculate, by induction, the coefficient $K_{\lambda\mu\nu}$ in the plethysm of S-functions $\{\lambda\}\otimes\{\mu\}=\sum K_{\lambda\mu\nu}\{\nu\}$ for all partitions of (λ) and (μ).

Tables showing the expansion of $\{\lambda\}\otimes\{\mu\}$ for all partitions up to a total degree of 18 in the resultant has been actually calculated and deposited with the Royal Society Mathematical Tables Committee. Copies of these tables are available on request and application should be made to The Royal Society.

It is known that the terms in the product of two tensors of type $\{\lambda_1\cdots\lambda_\gamma\}$ and $\{\mu_1\cdots\mu_\gamma\}$ respectively may correspond to any or all of the terms which appear in the product of these S-functions. The term of type $\{\lambda_1+\mu_1,\cdots,\lambda_\gamma+\mu_\gamma\}$, however, is the only term which is never zero. It is called the principal part of the product. A concomitant is only regarded as reducible if it is the principal part of a product of concomitants of lower degrees or is a linear combination of principal parts. If the number of parts in one partition is less than in the other partition zero parts are added to make them equal. This applies in every case throughout the paper.

THEOREM. *The principal parts of the products of terms in the expansion* $(\{\lambda_1\lambda_2,\cdots,\lambda_\gamma\}\otimes\{w_1w_2,\cdots,w_n\})(\{\mu_1\mu_2,\cdots,\mu_\gamma\}\otimes\{v_1v_2,\cdots,v_n)\}$ *appear as terms in the expansion of* $\{\lambda_1+\mu_1,\lambda_2+\mu_2,\cdots,\lambda_\gamma+\mu_\gamma\}\otimes\{\nu_1\nu_2,\cdots,\nu_n\}$ *whenever* $\chi_\rho^{w}\chi_\rho^{v}=\chi_\rho^{\nu}$ *for all classes.* χ_ρ^{w}, χ_ρ^{v}, *and* χ_ρ^{ν} *are the characters of the symmetric group of order* $n!$ *corresponding to the partitions* (w), (v), (ν) *of* n *and* (λ), (μ) *may be any other partitions.*

The theorem does not imply that the frequency of occurrence of a partition in $\{\lambda_1+\mu_1,\lambda_2+\mu_2,\cdots,\lambda_\gamma+\mu_\gamma\}\otimes\{\nu_1\nu_2,\cdots,\nu_n\}$ is at least as great as the number of ways in which it appears as a principal part of products of terms in $(\{\lambda_1\lambda_2\cdots\lambda_\gamma\}\otimes\{w_1w_2\cdots w_n\})\cdot(\{\mu_1\mu_2\cdots\mu_\gamma\}\otimes\{v_1v_2\cdots v_n\})$. Thus $(\{2\}\otimes\{2\})(\{2\}\otimes\{2\})=(\{4\}+\{2^2\})(\{4\}+\{2^2\})$ gives principal parts $\{8\}+2\{62\}+\{4^2\}$. But in $\{4\}\otimes\{2\}=\{8\}+\{62\}+\{4^2\}$ the coefficient of $\{62\}$ is unity. Each product $\{4\}\{2^2\}$ corresponds to the same coefficient. A coefficient greater than one can be assumed when these coefficients appear in the individual forms of

$$\{\lambda_1\lambda_2\cdots\lambda_\gamma\}\otimes\{w_1w_2\cdots w_n\}\quad\text{or}\quad\{u_1u_2\cdots u_\gamma\}\otimes\{v_1v_2\cdots v_n\}.$$

Thus the coefficient of $\{543^221\}$ in $\{321\}\otimes\{21\}$ is 60 and $\{1\}\otimes\{1^3\}=\{1^3\}$, thence the coefficient of $\{654321\}$ in $\{421\}\otimes\{21\}$ is at least 60. Careful analysis of the structure of the concomitant may indicate that a coefficient even higher than this might be inferred. This principal applies to the subsequent special cases.

(i) If $\{w_1w_2 \cdots w_n\} \equiv \{v_1v_2 \cdots v_n\} \equiv \{n\}$.

(ii) If $\{w_1w_2 \cdots w_n\} \equiv \{v_1v_2 \cdots v_n\} \equiv \{1^n\}$.

(iii) $\{w_1w_2 \cdots w_n\} \equiv \{n\}$ and $\{v_1v_2 \cdots v_n\} \equiv \{1^n\}$.

The previous cases have been proved elsewhere [5].

(iv) If $\{w_1w_2 \cdots w_n\} \equiv \{n\}$ then "The principal parts of the products of terms from the expressions $(\{\lambda_1\lambda_2 \cdots \lambda_\gamma\} \otimes \{n\})(\{\mu_1\mu_2 \cdots \mu_\gamma\} \otimes \{v\})$ appear as terms in the expansions of $\{\lambda_1+\mu_1, \lambda_2+\mu_2 \cdots \lambda_\gamma+\mu_\gamma\} \otimes \{v\}$ where (v) is a partition of (n)."

PROOF. Let there be n ground forms of type $\{\lambda_1\lambda_2 \cdots \lambda_\gamma\}$ and let the ith be expressed symbolically in terms of the symbols $\alpha_i, \alpha_i', \alpha_i'', \cdots$.

Let there be another set of n ground forms of type $\{\mu_1\mu_2 \cdots \mu_\gamma\}$ and express the ith symbolically in terms of the same symbols $\alpha_i, \alpha_i', \alpha_i'' \cdots$. The product of the symbolic expressions for the ith ground form in each set will be of the degree $(\lambda_1+\mu_1)$ in α_i, $(\lambda_2+\mu_2)$ in α_i', etc.; and may be interpreted as a ground form of type $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots, \lambda_\gamma+\mu_\gamma\}$.

Let ϕ be a symbolic expression giving a symmetric concomitant linear in each of the first set of ground forms and let ψ be a symbolic expression giving a concomitant of class $\{v\}$ [7] in the second set of ground forms. Then the product $\phi\psi$ may be interpreted as a concomitant of n ground forms of type $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots, \lambda_\gamma+\mu_\gamma\}$. But the effect of interchanging any two of these ground forms will leave the sign of ϕ unaltered while ψ will be altered in sign only if it is skew-symmetric in these two ground forms. Thus $\phi\psi$ will change sign for the interchange of any two ground forms in which ψ is skew-symmetric and will be unaltered in sign for the interchange of any two ground forms in which ψ is symmetric. Thus the concomitant $\phi\psi$ should be of class $\{v\}$. This proves the theorem.

(v) If $\{w_1w_2 \cdots w_n\} = \{1^n\}$ then "The principal parts of the products of individual terms in the expansion of $(\{\lambda_1\lambda_2 \cdots \lambda_\gamma\} \otimes \{1^n\})(\{\mu_1\mu_2 \cdots \mu_\gamma\} \otimes \{v\})$ appear as terms in the expansion of $\{\lambda_1+\mu_1, \cdots, \lambda_\gamma+\mu_\gamma\} \otimes \{v^*\}$ where (v^*) is conjugate to (v) a partition of n."

PROOF. Follow the same proof as in case (iv) but let ϕ be this time a symbolic expression giving an alternating concomitant linear in each of the first set of ground forms and ψ a symbolic expression giving a concomitant of class $\{v\}$ in the second set of ground forms. Then $\phi\psi$ may be interpreted as a concomitant of n ground forms of type $\{\lambda_1+\mu_1, \lambda_2+\mu_2, \cdots, \lambda_\gamma+\mu_\gamma\}$. But the effect of interchanging any 2 of these ground forms will be to change the sign of ϕ and that of ψ if it is skew-symmetric in these 2 ground forms. If ψ is symmetric in the 2 ground forms it will be unaltered in sign. Thus $\phi\psi$ will change sign for the interchange of any 2 ground forms in which ψ is symmetric and will be unaltered in sign if ψ is skew-symmetric in the 2 ground forms. Therefore the concomitant $\phi\psi$ should be of class $\{v^*\}$. This proves the theorem.

To illustrate the usefulness of these theorems the expansion of $\{41\}\otimes\{21\}$ has been computed and it was found that 181 terms could be deduced. Littlewood's third method leads to 47 other terms making a full expansion.

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