where \( C_r, C_f, C_o \) are certain positive constants, then it follows that along each direction there exists only one integral curve.

**References**


Translations of this article are available by writing to the Ordnance Research Laboratory, University Park, Pennsylvania. The article is also translated in American Mathematical Society Translations, ser. 2, vol. 1, 1955, pp. 239–252.

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**A DEVELOPMENT OF CARDINALS IN "THE CONSISTENCY OF THE CONTINUUM HYPOTHESIS"**

H D Sprinkle

In *The consistency of the continuum hypothesis* the axiom of choice is used to construct the theory of powers within that of cardinals. The main purpose here is to develop just the theory of cardinals without such an axiom. This work could be formalized with the use of the predicate calculus, but, as is done in the book, the proofs will be presented rather informally. Numbering of the definitions and theorems is intended to follow closely that of the numbering in the book.

**8.2** Dfn \( a \simeq a \cdot a \subseteq \text{On} \cdot (\beta) (\beta < a) \cdot \sim (\beta \simeq a) \cdot \text{On} = \text{On} \).

The existence of \( a \) follows from 7.7 and the unicity is evident. For \( Y = \alpha \) and \( Y = \text{On} \), \( Y \) is a normal operation, since \( X \subseteq Y \) if \( X \subseteq \text{On} \cdot (\alpha) \) \( (a \simeq Y) \cdot X \subseteq a \). Hence by MS taking \( B = \text{On} \) there exists a function \( 'Nc \), over \( \text{On} \) such that \( 'Nc'c = a \).

**8.20** Dfn \( 'Nc'c = a \cdot 'Nc'c \cdot \text{On} \).

**8.21** Dfn \( 'N = 'N('Nc) \).

**8.28** \( a \subseteq \beta \cdot a \subseteq \beta \).

**Proof.** \( \beta \simeq \beta \) so each element of \( \alpha \) is associated with an element of \( \beta \) by the similarity; i.e. \( (\exists x)(\alpha \simeq x \subseteq \beta) \). But \( x \) being a subset of \( \beta \) is

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1 A single bar placed over either a capital or lower case letter or expression has the same effect as a double bar would have in the book.
well-ordered by \( E \) and hence by 7.7 \((\exists h)(h \exists \text{om}_{EE}(\gamma, x)). \delta \leq h' \delta \)
for \( \delta \in \gamma \) by 7.611, and \( h' \delta \in x \subseteq \beta \) for \( \delta \in \gamma \). Hence \( \delta \in \gamma. \gamma \leq h' \delta \subseteq \beta \),
that is \( \gamma \subseteq \beta \). \( \alpha \approx x \gg \gamma \subseteq \beta \) so \( \omega = \gamma \leq \beta \).

If \( \alpha \approx x \subseteq \beta \) and \( \beta \approx y \subseteq \alpha \) then \( \alpha \approx \beta \) follows as a consequence, since
by 7.7 and 7.611 there are \( \gamma \) and \( \delta \) such that \( x \approx \gamma \subseteq \beta \) and \( y \approx \delta \subseteq \alpha \).
This means that \( \alpha \approx y \approx \beta \approx \gamma \approx \delta \leq \alpha \) so that \( \alpha = \beta \) or \( \alpha \approx \beta \). 8.28 is used to help prove \( M(\exists x) \) in 8.292.

8.29

\[(\alpha)(\exists \beta)(\beta > \alpha \cdot \approx (\beta \approx \alpha)).\]

**Proof.** Let \( \gamma \in K(\beta) \equiv (\exists \alpha, x)(\alpha = y \cdot \beta \exists \alpha \cdot \alpha \text{ is isomorphic to } \beta \)
with respect to \( E \) and \( x \). Existence follows from M3 and uniqueness
is guaranteed by the axiom of extensionality. Intersect each of the
well-orderings of \( \beta \) with \( (\beta + 1) \times (\beta \times 1) \), then it is an element of
\( \Psi((\beta + 1) \times (\beta + 1)) \). Hence the class of all such is contained in
\( \Psi((\beta + 1) \times (\beta + 1)) \); therefore, this class is a set by 5.12, 5.121, and
5.18. It should be noticed that \( x \mid ((\beta + 1) \times (\beta + 1)) \) is no essential
restriction on a well-ordering \( x \) of \( \beta \). With each well-ordering of \( \beta \) is
associated a unique ordinal number \( \alpha \) by 7.62 and 7.7. This class of
ordinals is a set by 5.1; i.e., \( M(\exists K(\beta)) \). Since \( M(\exists K(\beta)) \) and by M5
there is a function \( K \circ \beta = K(\beta) \cdot K \circ \beta \cdot On \). \( M(\exists K(\beta + 1)) \) by 5.1.
Since \( \alpha \in K(\alpha) \) it is an element of \( \mathcal{E}(K \circ(\alpha + 1)) \), so \( \mathcal{E}(K \circ(\alpha + 1)) + 1 > \alpha \), and \( \mathcal{E}(K \circ(\alpha + 1)) + 1 \) is also an ordinal number. If \( \mathcal{E}(K \circ(\alpha + 1)) + 1 \approx \alpha \) then \( \mathcal{E}(K \circ(\alpha + 1)) + 1 \in K(\alpha) \) as the similarity of \( \alpha \)
with \( \mathcal{E}(K \circ(\alpha + 1)) + 1 \) would impose a well-ordering \( x \) on \( \alpha \). Therefore
\( \mathcal{E}(K \circ(\alpha + 1)) + 1 \in \mathcal{E}(K \circ(\alpha + 1)) \), which is a contradiction.
Take \( \mathcal{E}(K \circ(\alpha + 1)) + 1 \) as a \( \beta \) in the theorem.

8.291 **Dfn** \( x \in \exists (\alpha) \equiv (\exists \alpha, \beta)(x = \alpha \cdot \beta \in A : \beta = \beta \approx \alpha). \)
M3 gives existence, and uniqueness followed by the axiom of extensionality.

8.292 \( M(\exists x) \).

**Proof.** \( \exists x \equiv (x \cdot On) \). \( \mathcal{E}(x \cdot On) \) is an ordinal number since
\( (x \cdot On) \) is a set, and so \( (\exists \beta)(\beta > \mathcal{E}(x \cdot On) \cdot \approx (\beta \approx \mathcal{E}(x \cdot On))). \)
\( \beta \in \exists (\mathcal{E}(x \cdot On)) \), for if it were \( \beta \approx \gamma \) (where \( \gamma \in \mathcal{E}(x \cdot On) \cdot \gamma \in (N) \).
This would mean that \( \gamma < \mathcal{E}(x \cdot On) < \beta \) and \( \gamma = \beta \), but
\( \mathcal{E}(x \cdot On) \leq \beta \),
a contradiction as \( \approx (\beta \approx \mathcal{E}(x \cdot On)). \beta > \alpha \), for \( \alpha \in \exists (\mathcal{E}(x \cdot On)) \), since
\( \beta \in \exists (\mathcal{E}(x \cdot On)) \), and since \( \exists (\mathcal{E}(x \cdot On)) \) contains all cardinals less
than any given one in it. \( \mathfrak{B}(x) = \mathfrak{B}(x \cdot \text{On}) \subseteq \mathfrak{B}(\mathfrak{C}(x \cdot \text{On})) \subseteq \mathfrak{P} \), so \( \mathfrak{M}(\mathfrak{B}(x)) \) by 5.12.

8.33 \( \mathfrak{P}(\mathfrak{N}) \).

**Proof.** Let \( m \subseteq \mathfrak{N} \) and consider \( \mathfrak{C}(\mathfrak{B}(m)) \). Since \( \mathfrak{B}(m) \) is a set, \( \mathfrak{C}(\mathfrak{B}(m)) \) is a set, and hence an ordinal number. \( \mathfrak{C}(\mathfrak{B}(m)) + 1 > \alpha \), for \( \alpha \in \mathfrak{B}(m) \); therefore,

\[
\mathfrak{S}(\mathfrak{B}(m)) + 1
\]

is greater than any cardinal in \( m \). So \( m \not\subseteq \mathfrak{N} \).

8.54 **Dfn** \( \mathfrak{N}' = \mathfrak{N} - \omega \).

8.55 \( \mathfrak{N}' \subseteq \text{On} \).

8.56 \( \mathfrak{N}' \) is isomorphic to \( \text{On} \) with respect to \( E \).

**Proof.** Since \( \omega \) is a set, \( \mathfrak{P}(\mathfrak{N}') \). Moreover, any proper section of \( \mathfrak{N}' \) is generated by an \( \alpha \in \mathfrak{N}' \); hence \( \subseteq \alpha \), so a set by 5.12. 7.7 gives the result. The isomorphism from \( \text{On} \) to \( \mathfrak{N}' \) is denoted by \( \mathfrak{N} \); i.e.,

8.57 **Dfn** \( \mathfrak{N} \cong \text{om}_E(\text{On}, \mathfrak{N}') \).

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