where C_ρ, C_f, C_0 are certain positive constants, then it follows that along each direction there exists only one integral curve.

References

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A DEVELOPMENT OF CARDINALS IN "THE CONSISTENCY OF THE CONTINUUM HYPOTHESIS"

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In *The consistency of the continuum hypothesis* the axiom of choice is used to construct the theory of powers within that of cardinals. The main purpose here is to develop just the theory of cardinals without such an axiom. This work could be formalized with the use of the predicate calculus, but, as is done in the book, the proofs will be presented rather informally. Numbering of the definitions and theorems is intended to follow closely that of the numbering in the book.

'8.2 Dfn $\alpha \simeq \bar{\alpha} \cdot \bar{\alpha} \in On \cdot (\beta)(\beta < \bar{\alpha} . \supset . \sim(\beta \simeq \alpha)) \cdot On = On.$

The existence of $\bar{\alpha}$ follows from 7.7 and the unicity is evident.[1] For $Y=\alpha$ and $Y=On$, $\overline{Y}$ is a normal operation, since $X \in \overline{Y} . \equiv : X \in On \cdot (\alpha)(\alpha \simeq' Y . \supset . X \in \alpha)$. Hence by M5 taking $B=On$ there exists a function $'Nc$, over On such that $'Nc`\alpha = \bar{\alpha}$.

'8.20 Dfn $'Nc`\alpha = \bar{\alpha} \cdot 'Nc\mathfrak{Fn}On.$

'8.21 Dfn $'N = \mathfrak{W}('Nc).$

'8.28 $\alpha \subseteq \beta . \supset . \bar{\alpha} \leqq \bar{\beta}.$

Proof. $\beta \simeq \bar{\beta}$ so each element of α is associated with an element of $\bar{\beta}$ by the similarity; i.e. $(\exists x)(\alpha \simeq x \subseteq \bar{\beta})$. But x being a subset of $\bar{\beta}$ is

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[1] A single bar placed over either a capital or lower case letter or expression has the same effect as a double bar would have in the book.

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well-ordered by E and hence by 7.7 $(\exists h)(h\mathfrak{Isom}_{EE}(\gamma, x))$. $\delta \leqq h`\delta$ for $\delta \in \gamma$ by 7.611, and $h`\delta \in x \subseteq \bar{\beta}$ for $\delta \in \gamma$. Hence $\delta \in \gamma . \supset . \delta \leqq h`\delta \in \bar{\beta}$, that is $\gamma \subseteq \bar{\beta}$. $\alpha \simeq x \simeq \gamma \subseteq \bar{\beta}$ so $\bar{\alpha} = \bar{\gamma} \leqq \gamma \leqq \bar{\beta}$.

If $\alpha \simeq x \subseteq \beta$ and $\beta \simeq y \subseteq \alpha$ then $\alpha \simeq \beta$ follows as a consequence, since by 7.7 and 7.611 there are γ and δ such that $x \simeq \gamma \subseteq \beta$ and $y \simeq \delta \subseteq \alpha$. This means that $\alpha \simeq \gamma$ and $\beta \simeq \delta$ or $\bar{\alpha} = \bar{\gamma} \leqq \bar{\beta}$ and $\bar{\beta} = \bar{\delta} \leqq \bar{\alpha}$ so that $\bar{\alpha} = \bar{\beta}$ or $\alpha \simeq \beta$. '8.28 is used to help prove $\mathfrak{M}(\mathfrak{Z}(x))$ in 8.292.

8.29 $$(\alpha)(\exists \beta)(\beta > \alpha \cdot \sim(\beta \simeq \alpha)).$$

Proof. Let $y \in K(\beta) . \equiv . (\exists \alpha, x)(\alpha = y \cdot \beta \mathfrak{W}\mathfrak{e} x \cdot \alpha$ is isomorphic to β with respect to E and x). Existence follows from M3 and uniqueness is guaranteed by the axiom of extensionality. Intersect each of the well-orderings of β with $(\beta \dot{+} 1) \times (\beta \times 1)$, then it is an element of $\mathfrak{P}((\beta \dot{+} 1) \times (\beta \dot{+} 1))$. Hence the class of all such is contained in $\mathfrak{P}((\beta \dot{+} 1) \times (\beta \dot{+} 1))$; therefore, this class is a set by 5.12, 5.121, and 5.18. It should be noticed that $x | ((\beta \dot{+} 1) \times (\beta \dot{+} 1))$ is no essential restriction on a well-ordering x of β.[2] With each well-ordering of β is associated a unique ordinal number α by 7.62 and 7.7. This class of ordinals is a set by 5.1; i.e., $\mathfrak{M}(K(\beta))$. Since $\mathfrak{M}(K(\beta))$ and by M5 there is a function $Ko`\beta = K(\beta) \cdot Ko \mathfrak{F}\mathfrak{n} On$. $\mathfrak{M}(Ko``(\beta \dot{+} 1))$ by 5.1. Since $\alpha \in K(\alpha)$ it is an element of $\mathfrak{S}(Ko``(\alpha \dot{+} 1))$, so $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1 > \alpha$, and $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1$ is also an ordinal number. If $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1 \simeq \alpha$ then $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1 \in K(\alpha)$ as the similarity of α with $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1$ would impose a well-ordering $x$ on $\alpha$. Therefore $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1 \in \mathfrak{S}(Ko``(\alpha \dot{+} 1))$, which is a contradiction. Take $\mathfrak{S}(Ko``(\alpha \dot{+} 1)) \dot{+} 1$ as a β in the theorem.

8.291 Dfn $x \in \mathfrak{Z}(A) . \equiv . (\exists \alpha, \beta)(x = \alpha \cdot \beta \in A \cdot \beta = \bar{\beta} \cdot \beta \simeq \alpha)$.

M3 gives existence, and uniqueness followed by the axiom of extensionality.

8.292 $$\mathfrak{M}(\mathfrak{Z}(x)).$$

Proof. $\mathfrak{Z}(x) = \mathfrak{Z}(x \cdot On)$. $\mathfrak{S}(x \cdot On)$ is an ordinal number since $(x \cdot On)$ is a set, and so $(\exists \beta)(\beta > \mathfrak{S}(x \cdot On) \cdot \sim(\beta \simeq \mathfrak{S}(x \cdot On)))$. $\beta \notin \mathfrak{Z}(\mathfrak{S}(x \cdot On))$, for if it were $\beta \simeq \gamma$ (where $\gamma \in \mathfrak{S}(x \cdot On) \cdot \gamma \in ' N$). This would mean that $\gamma < \mathfrak{S}(x \cdot On) < \beta$ and $\bar{\gamma} = \bar{\beta}$, but

$$\overline{\mathfrak{S}(x \cdot On)} \leqq \bar{\beta},$$

a contradiction as $\sim(\beta \simeq \mathfrak{S}(x \cdot On))$. $\beta > \alpha$, for $\alpha \in \mathfrak{Z}(\mathfrak{S}(x \cdot On))$, since $\beta \in \mathfrak{Z}(\mathfrak{S}(x \cdot On))$, and since $\mathfrak{Z}(\mathfrak{S}(x \cdot On))$ contains all cardinals less

[2] The confirming relation is denoted by "$|$".

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than any given one in it. $\mathfrak{Z}(x)=\mathfrak{Z}(x\cdot On)\subseteq\mathfrak{Z}(\mathfrak{S}(x\cdot On))\subseteq\beta$, so $\mathfrak{Mr}(\mathfrak{Z}(x))$ by 5.12.

'8.33 $$\mathfrak{Pr}('N).$$

Proof. Let $m\subseteq'N$ and consider $\mathfrak{S}(\mathfrak{Z}(m))$. Since $\mathfrak{Z}(m)$ is a set, $\mathfrak{S}(\mathfrak{Z}(m))$ is a set, and hence an ordinal number. $\mathfrak{S}(\mathfrak{Z}(m))\dotplus 1>\alpha$, for $\alpha\in\mathfrak{Z}(m)$; therefore,

$$\overline{S(\mathfrak{Z}(m))+1}$$

is greater than any cardinal in m. So $m\neq'N$.

'8.54 $$\text{Dfn } 'N' = 'N - \omega.$$

'8.55 $$'N' \subseteq On.$$

'8.56 $'N'$ is isomorphic to On with respect to E.

Proof. Since ω is a set, $\mathfrak{Pr}('N')$. Moreover, any proper section of $'N'$ is generated by an $\alpha\in'N'$; hence $\subseteq\alpha$, so a set by 5.12. 7.7 gives the result. The isomorphism from On to $'N'$ is denoted by $'\aleph$; i.e.,

'8.57 $$\text{Dfn } '\aleph\,\mathfrak{Isom}_{EE}(On, 'N').$$

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