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COMPONENT SUBSETS OF THE FREE LATTICE ON n GENERATORS

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1. **Introduction.** In the free lattice on n generators, $\mathrm{FL}(n)$, the *components* of a word (lattice polynomial) are defined recursively by: (i) the only component of a generator is itself, and (ii) if $W \equiv A \cup B$ (or $A \cap B$) the components of W are W, A, and B and their components.[1] A *component subset*, P, of $\mathrm{FL}(n)$ is a collection of words in $\mathrm{FL}(n)$ with the following property: if a word belongs to P then so do all its components.[2] A component subset of $\mathrm{FL}(n)$ may be partially ordered in a natural way: $A \geqq B$ if and only if $A \geqq B$ in $\mathrm{FL}(n)$. Clearly if $W \equiv A \cup B$ (or $A \cap B$) belongs to a component subset, P, W appears in P as the l.u.b. (or g.l.b.) of A and B under the ordering $(\geqq)$. Thus it is natural to say that a component subset P is generated by the generators of $\mathrm{FL}(n)$ which appear in the words belonging to P. This notion is used here to prove:

Theorem 1.[3] *Given any two words, unequal in* $\mathrm{FL}(n)$, *there exists a finite homomorphic image of* $\mathrm{FL}(n)$ *in which their images are distinct.*

Theorem 2.[4] *Any lattice possessing a countable number of generators*

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[1] $(\equiv)$ denotes logical identity.

[2] The author is indebted to R. P. Dilworth for suggesting the definition of component subset and its application in the proof of Theorem 2.

[3] Theorem 1 was conjectured by Marshall Hall, Jr.

[4] Theorem 2 is due to Sorkin [4] where a sketch of a proof is given.

can be embedded in a lattice with 3 *generators.*

From the construction given in the proof of Theorem 2, it follows that a finite lattice can be embedded in a lattice with 3 generators.

2. **Proof of Theorem 1.** Let A and B be two unequal words in FL(n). Consider the component subset P of FL(n) formed by $A \cup B$ and its components. Let $x_1, \cdots, x_k$ be the generators of FL(n) appearing in $A \cup B$, renumbering if necessary. As indicated in the introduction, P is a partially ordered set generated by $x_1, \cdots, x_k$ in which A and B appear as distinct elements. Using the MacNeille completion by cuts [3], P may be completed to a lattice $L(P)$. $L(P)$ is an embedding of P which preserves all l.u.b.'s and g.l.b.'s which exist in P. In the embedding, let $X \in P$ correspond to $X' \in L(P)$. An element W of P, not a generator, has the form $W \equiv X \cup Y$ (or $X \cap Y$). As remarked in the introduction, W is the l.u.b. (or g.l.b.) of X and Y in P under ($\geqq$). Hence in $L(P)$, $W' = X' \cup Y'$ (or $X' \cap Y'$). Moreover, the elements added in the completion are just the l.u.b.'s and g.l.b.'s of elements in P, so that $L(P)$ is generated by the k generators, $x_1', \cdots, x_k'$. Since P is embedded in $L(P)$, $A' \neq B'$ and, by the construction, $L(P)$ is finite since P is finite.

It remains to be shown that there is a homomorphism mapping FL(n) onto $L(P)$ and $A \rightarrow A'$ and $B \rightarrow B'$. The desired homomorphism is generated by $x_i \rightarrow x_i'$, $i = 1, \cdots, k-1$; $x_{k+j} \rightarrow x_k'$, $j = 0, \cdots, n-k$. This mapping can be extended to a homomorphism of FL(n) onto L(P),[5] and since unions and intersections are preserved we have, proceeding inductively, that if $W \in P$, $W \equiv X \cup Y$, and $X \rightarrow X'$ and $Y \rightarrow Y'$, then $W \equiv X \cup Y \rightarrow W' = X' \cup Y'$, and dually. In particular, $A \rightarrow A'$ and $B \rightarrow B'$.

3. **Quotients in** FL(3).

LEMMA 1. *In* FL(3) *with generators a, b, c the six quotients given by* $(a \cup b) \cap (a \cup c)/a \cup (b \cap c)$, *its cyclical forms and their duals, and the quotient* $(a \cup b) \cap (a \cup c) \cap (b \cup c)/(a \cap b) \cup (a \cap c) \cup (b \cap c)$ *are mutually exclusive and contain every word whose length, when written in canonical form*[6] *is* $\geqq 4$.

PROOF. These quotients are mutually exclusive because under the homomorphism generated by the distributive law all elements of a particular quotient have a common image and elements in different quotients have distinct images. The second part of the lemma follows

[5] See, for example, Birkhoff [1, p. 30].

[6] See Whitman [5] for the definition of these terms.

from Whitman [6], Lemma 1.1 and Theorem 3.

The following table is constructed as a guide for future calculations. Let

$$\begin{aligned}
\mathrm{I}&: (a \cup b) \cap (a \cup c)/a \cup (b \cap c)\\
\mathrm{II}&: (a \cup b) \cap (b \cup c)/b \cup (a \cap c)\\
\mathrm{III}&: (a \cup c) \cap (b \cup c)/c \cup (a \cap b)\\
\mathrm{IV}&: a \cap (b \cup c)/(a \cap b) \cup (a \cap c)\\
\mathrm{V}&: b \cap (a \cup c)/(a \cap b) \cup (b \cap c)\\
\mathrm{VI}&: c \cap (a \cup b)/(a \cap c) \cup (b \cap c)\\
\mathrm{VII}&: (a \cup b) \cap (a \cup c) \cap (b \cup c)/(a \cap b) \cup (b \cap c) \cup (a \cap c).
\end{aligned}$$

In Table 1, the entry above the main diagonal in row i, column j is $x \cup y$ or the quotient to which $x \cup y$ belongs when x is the element, or belongs to the quotient, heading row i and y is the element, or belongs to the quotient, heading column j. Similarly, $x \cap y$ is entered below the main diagonal.

TABLE 1

$\cup$ \ $\cap$	a	b	c	I	II	III	IV	V	VI	VII
a	a	$a\cup b$	$a\cup c$	I	$a\cup b$	$a\cup c$	a	I	I	I
b	$a\cap b$	b	$b\cup c$	$a\cup b$	II	$b\cup c$	II	b	II	II
c	$a\cap c$	$b\cap c$	c	$a\cup c$	$b\cup c$	III	III	III	c	III
I	a	V	VI	I	$a\cup b$	$a\cup c$	I	I	I	I
II	IV	b	VI	VII	II	$b\cup c$	II	II	II	II
III	IV	V	c	VII	VII	III	III	III	III	III
IV	IV	$a\cap b$	$a\cap c$	IV	IV	IV	IV	VII	VII	VII
V	$a\cap b$	V	$b\cap c$	V	V	V	$a\cap b$	V	VII	VII
VI	$a\cap c$	$b\cap c$	VI	VI	VI	VI	$a\cap c$	$b\cap c$	VI	VII
VII	IV	V	VI	VII	VII	VII	IV	V	VI	VII

LEMMA 2. *If $X \in \mathrm{I}$ and X has a canonical form $X \equiv a \cup Y$, then*

$$(1) \qquad Y \in \mathrm{VI} \textit{ implies } X \cap b \textit{ is canonical form}$$

and

$$(2) \qquad Y \in \mathrm{V} \textit{ implies } X \cap c \textit{ is canonical form.}$$

PROOF. (1) shall be proved. The proof of (2) follows by interchanging b and c. First observe that $X \cap b \in \mathrm{V}$. Let $X \cap b = W$ where W is in canonical form.

CASE 1. $W \equiv \bigcup_i W_i$. By Corollary 1.2, Whitman [6], $X \cap b = X$ or b. Thus $X \geqq b$ or $b \geqq X$, but neither alternative is possible as $X \in \mathrm{I}$.

CASE 2. $W \equiv \bigcap_i W_i = X \cap b$. By the dual to (19), Whitman [5], $W_i \geqq X$ or b, for all i. If $W_i \geqq X$ for all i, then $X \cap b \geqq X \geqq X \cap b$. As in Case 1, this is impossible. Hence $W_i \geqq b$ for some i. On the other hand $b \geqq \bigcap_i W_i$, hence $b \geqq W_j$ for some j. By the assumed canonicity of W, it follows that $i = j$, and only one such index exists; let $i = 1$. Hence for $i > 1$, $W_i \geqq X$. Thus $X \equiv a \cup Y \geqq b \cap \bigcap_{i>1} W_i \equiv W$. By (17) Whitman [5], this holds if and only if one of the following hold:

(1) $a \geqq W$, (2) $Y \geqq W$, (3) $X \geqq b$, (4) $X \geqq W_j$, for some $j \geqq 2$. (1) can't hold as $W \in \mathrm{V}$, (2) can't hold as $Y \in \mathrm{VI}$, $W \in \mathrm{V}$, and (3) can't hold as $X \in \mathrm{I}$. But if (4) holds, and since it is the only alternative left it must hold for some $j \geqq 2$, $W_i \geqq X \geqq W_j$, for all $i \geqq 2$. Thus, by the canonicity of W, $i = j$ and only one such index exists. Thus $\bigcap_i W_i \equiv X \cap b$ is the canonical form of W. Cyclical and dual lemmas hold.

4. **Proof of Theorem 2.** Let L have n generators g_i, $i = 1, 2, 3, \cdots$. In FL(3) consider the generators of a sublattice isomorphic to FL($2n$). In particular, select those generators, a_m, given by Dilworth [2]. Let F denote the component subset of FL(3) formed by the a_m, $m = 1, 2, \cdots$, and their components. Thus the elements of F and the quotients to which they belong are $X_0 = a$, and for $m = 1, 2, \cdots$:

$$\alpha_m \equiv c \cap X_{m-1} \in \mathrm{VI}(\alpha_1 = a \cap c),$$

$$\alpha_{-m} \equiv c \cup X_{-m+1} \in \mathrm{III}(\alpha_{-1} = a \cup c),$$

$$\beta_m \equiv b \cup \alpha_m \in \mathrm{II}, \qquad \beta_{-m} \equiv b \cap \alpha_{-m} \in \mathrm{V},$$

$$\gamma_m \equiv a \cap \beta_m \in \mathrm{IV}, \qquad \gamma_{-m} \equiv a \cup \beta_{-m} \in \mathrm{I},$$

$$\delta_m \equiv c \cup \gamma_m \in \mathrm{III}, \qquad \delta_{-m} \equiv c \cap \gamma_{-m} \in \mathrm{VI},$$

$$\epsilon_m \equiv b \cap \delta_m \in \mathrm{V}, \qquad \epsilon_{-m} \equiv b \cup \delta_{-m} \in \mathrm{II},$$

$$X_m \equiv a \cup \epsilon_m \in \mathrm{I}, \qquad X_{-m} \equiv a \cap \epsilon_{-m} \in \mathrm{IV},$$

$$\mu_m \equiv X_m \cap \lambda_{-m} \in \mathrm{VII}, \qquad \lambda_{-m} \equiv c \cup X_{-m} \in \mathrm{III}(= \alpha_{-m-1}),$$

$$a_m \equiv b \cup \mu_m \in \mathrm{II}.$$

LEMMA 3. *The elements of F are in canonical form.*

PROOF. By applying Lemma 2 and its cyclical and dual forms it is clear that $\alpha_m, \beta_m, \gamma_m, \delta_m, \epsilon_m, X_m$ and λ_m, $m = \pm 1, \pm 2, \cdots$, are in canonical form. The canonicity of μ_m and a_m is established by verifying Whitman's criteria, [6, Corollary 1.1].

For μ_m, it is clear that X_m and λ_{-m} are in canonical form and noncomparable. $X_m \cap \lambda_{-m} \equiv (a \cup \epsilon_m) \cap (X_{-m} \cup c)$ and since $a \ngeq \mu_m$, $\epsilon_m \ngeq \mu_m$, $X_{-m} \ngeq \mu_m$, and $c \ngeq \mu_m$, it follows that μ_m must be in canonical form.

For $a_m \equiv b \cup \mu_m \equiv b \cup (X_m \cap \lambda_{-m})$, it is clear that b and μ_m are in canonical form and noncomparable and, since $X_m \ngeq a_m$ and $\lambda_{-m} \ngeq a_m$, hence a_m is in canonical form.

Proceeding with the proof of Theorem 2, the elements of L and F are combined into a single set $P = [L, F]$ which is ordered as follows:

DEFINITION 1. For X and $Y \in P$, $X \supseteq Y$ if and only if one or more of the following hold:

(1) $X, Y \in L$ and $X \geqq Y$ in L.

(2) $X, Y \in F$ and $X \geqq Y$ in FL(3).

(3) $X = g_m$ and $Y = a_{2m-1}$ or a_{2m}, $m = 1, 2, \cdots$.

(4) $X \in L$, $Y \in F$ and $X \geqq U_1 \cap \cdots \cap U_r$ in L and for every i there is a j such that $U_i \geqq g_j$ and a_{2j-1} or $a_{2j} \geqq Y$ in FL(3).

The proof of Theorem 2 rests on the following lemmas.

LEMMA 4. ($\supseteq$) *is a partial ordering on P.*

LEMMA 5. ($\supseteq$) *preserves unions and intersections existing in L.*

LEMMA 6. *No element of F contains a_m except a_m.*

LEMMA 7. *g_i is the l.u.b. under $\supseteq$ of a_{2i-1} and a_{2i}.*

LEMMA 8. ($\supseteq$) *preserves unions and intersections in F which hold in* FL(3).

The proof of the Theorem 2 now follows directly. Using the MacNeille completion by cuts [3], complete the partially ordered set $P = [L, F, \supseteq]$ to a lattice $L(P)$. Since by Lemma 5, ($\supseteq$) preserves unions and intersections in L, $L(P)$ is an embedding for L. Since a, b, c generate F and, by Lemma 7, g_m is the l.u.b. of a_{2m-1} and a_{2m} in P, and since, by Lemma 8, ($\supseteq$) preserves unions and intersections in F, a, b, c, generate P and hence $L(P)$ also. Finally, P is finite if L is finite. Thus the following corollary is proved.

COROLLARY. *Any finite lattice can be embedded in a finite lattice with three generators.*

The proofs of Lemmas 4 and 5 are left to the reader. Lemma 7 follows immediately from Lemma 6 and Definition 1. Lemmas 6 and 8 will be proved in detail.

PROOF OF LEMMA 6. Since $a_m \in$ II, the only candidates for X such that $X \geqq a_m$ are ϵ_{-r}, β_r, a_r. $a_r \geqq a_m$ if and only if $r=m$, since the a_m are noncomparable. Suppose that $\epsilon_{-r} \geqq a_m$. This holds if and only if $\epsilon_{-r} \geqq \mu_m$. This last relation holds in FL(3) if and only if one of the following hold:

$$(1)\ \epsilon_{-r} \geqq X_m, \qquad (2)\ \epsilon_{-r} \geqq \lambda_{-m}, \qquad (3)\ b \geqq \mu_m, \qquad (4)\ \delta_{-r} \geqq \mu_m.$$

However none of these can hold since the elements belong to the wrong quotients—see Table 1.

Suppose that $\beta_r \geqq a_m$. This holds if and only if $\beta_r \geqq \mu_m$. Again one of four possibilities must hold:

$$(1)\quad \beta_r \geqq X_m, \qquad (2)\quad \beta_r \geqq \lambda_{-m}, \qquad (3)\quad b \geqq \mu_m, \qquad (4)\quad \alpha_r \geqq \mu_m.$$

As before, none of these can be true.

PROOF OF LEMMA 8. PART 1. Suppose $W, X, Y \in F$ and $W = X \cap Y$ in FL(3). Then X and $Y \geqq W$, hence X and $Y \supseteq W$ in P. Thus W is a lower bound for X and Y. Let X and $Y \supseteq Z$ in P. Since X and $Y \in F$, $Z \in F$ also, as no element of F can contain ($\supseteq$) an element of L, by definition. Thus X and $Y \geqq Z$, whence $W = X \cap Y \geqq Z$ and thus $W \supseteq Z$.

PART 2. Suppose $W, X, Y \in F$ and $W = X \cup Y$ in FL(3). By the dual of the first argument in Part 1, W is an upper bound for X and Y under $\supseteq$. Suppose $Z \supseteq X$ and Y. If $Z \in F$ the dual of the second argument in Part 1 shows that $Z \supseteq W$. Suppose that $Z \in L$. Now $Z \supseteq X$, Y and X, $Y \in F$ imply that there exist indices r and s such that $a_r \geqq X$ and $a_s \geqq Y$. Thus X and Y must come from quotients II, IV, V, VI, VII, or $=b$. From a study of Table 1, $X \cup Y$ must then belong to quotients II, VII, IV (if and only if X and $Y \in$ IV), V (if and only if X and $Y \in$ V), VI (if and only if X and $Y \in$ VI), or $=b$.

If X and Y both belong to V or $X \cup Y = b$, then every $a_m \geqq X \cup Y$ and in this case $Z \supseteq X \cup Y$.

When $W = X \cup Y \in$ IV, VI, or VII, it will be shown that $W = X \cup Y = X$ or Y and thus $Z \supseteq X$ and Y implies trivially that $Z \supseteq W = X \cup Y$. Since $W = X \cup Y \in$ IV, VI, or VII, W must have the canonical form $p \cap q$ where p and q are generators or intersections. Now $W = X \cup Y \equiv \bigcup_i T_i$ where each T_i appears in F as a generator or as an intersection. From the dual of Corollary 1.2, Whitman [6], $W = X \cup Y \equiv \bigcup_i T_i = T_k$ for some index k. Thus since X and Y are unions of certain of the T_i, X or $Y \geqq T_k$. But $T_k \geqq X$ and Y. Thus $W = T_k = X$ or Y.

When $W = X \cup Y \in$ II, W must have the canonical form $b \cup p$ where p is an intersection. Thus $W = X \cup Y \equiv \bigcup_i T_i$ where each T_i appears in F as a generator or as an intersection. From (19) Whitman [5], $b \leqq T_h$ and $p \leqq T_k$ for some h and k. Thus $W \equiv b \cup p \leqq T_h \cup T_k \leqq \bigcup_i T_i = X \cup Y = W$. Since X and Y are unions of certain of the T_i, X or $Y \geqq T_h$ and X or $Y \geqq T_k$. If either X or Y contains both T_h and T_k, then X and Y are comparable in F so that $Z \supseteq X$ and Y implies trivially that $Z \supseteq W = X \cup Y$.

Suppose then that $X \geqq T_h \geqq b$ and $Y \geqq T_k \geqq p$. Thus $Z \supseteq Y$ implies $Z \supseteq p$. Now $Z \supseteq Y$ and $Z \in L$ implies that $Z \geqq u_1 \cap \cdots \cap u_r$ in L where for each index i there exists a j such that $u_i \geqq g_j$ and either a_{2j-1} or $a_{2j} \geqq Y \geqq p$ in FL(3). But every $a_m \geqq b$. Thus either a_{2j-1} or $a_{2j} \geqq b \cup p$. Thus $Z \supseteq W = X \cup Y = b \cup p$.

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