THE COMPLEMENT OF A FINITELY GENERATED DIRECT SUMMAND OF AN ABELIAN GROUP

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1. In his recent monograph on abelian groups [1], Kaplansky raises the following question:

If $F$ is a finitely generated abelian group and $G$, $H$ are any abelian groups such that $F \oplus G \cong F \oplus H$, are $G$ and $H$ isomorphic?

We shall answer this question affirmatively. In the first place we can reduce the problem to the case where $F$ is cyclic of infinite or prime power order. For suppose that the answer has been obtained in this case, and let $F$ be any finitely generated abelian group. Then $F$ is a direct sum of a finite number of cyclic groups, each of infinite or prime power order, say

$$F = F_1 \oplus F_2 \oplus \cdots \oplus F_k,$$

and

$$F_1 \oplus \cdots \oplus F_k \oplus G \cong F_1 \oplus \cdots \oplus F_k \oplus H.$$

We use induction on $k$. By the result for the cyclic case, we may cancel $F_i$ and obtain

$$F_2 \oplus \cdots \oplus F_k \oplus G \cong F_2 \oplus \cdots \oplus F_k \oplus H,$$

and therefore $G \cong H$, by induction.

2. It remains to deal with the cyclic case. By identifying $F \oplus G$ and $F \oplus H$ we may restate the assertion as follows: If $E$ is an abelian group which may be written as a direct sum in two ways, $E = A \oplus G = B \oplus H$, where $A$ and $B$ are cyclic subgroups of $E$ of the same order $\infty$ or $p^n$, then $G \cong H$.

3. We first dispose of the case where $A$ and $B$ are infinite. Let $G \cap H = D$, then $G/D \cong G/G \cap H \cong (G + H)/H \triangleq$ subgroup of $E/H \cong B$. Thus $G/D$ is infinite cyclic or 0. Similarly for $H/D$. If $G/D$ and $H/D$ are both zero, then $G = D = H$, so suppose that $G/D$ is infinite cyclic. Choose $u \in G$ such that the residue-class $D + u$ generates $G/D$, and let $U$ be the subgroup of $G$ generated by $u$. Then $G = U \oplus D$, as is
easily seen. Thus \( E = A \oplus U \oplus D = B \oplus H \), and \( D \subseteq H \). Taking quotients by \( D \) we obtain \( A \oplus U \cong B \oplus H / D \), hence \( H / D \) is infinite cyclic; any representative in \( H \) of a generator mod \( D \) generates an infinite cyclic group \( V \) which satisfies \( H = V \oplus D \). Thus \( G = U \oplus D \cong V \oplus D = H \).

4. Now suppose that \( A \) and \( B \) are finite, of order \( p^n \) say, and are generated by \( a \) and \( b \) respectively. We show first that there exists an element \( u \) of order \( p^n \) such that no nonzero multiple of \( u \) belongs to \( G \) or \( H \). If neither \( a \) nor \( b \) satisfies this condition, then since no nonzero multiple of \( a \) lies in \( G \), we must have \( p^{n-1}a \in H \), and similarly \( p^{n-1}b \in G \). Put \( u = a + b \), then \( p^nu = 0 \), while \( p^{n-1}u = p^{n-1}a \neq 0 \) (mod \( G \)) and \( p^{n-1}u = p^{n-1}b \neq 0 \) (mod \( H \)). Thus there is always such an element \( u \). Let \( U \) be the subgroup generated by \( u \). Then from the definition, \( U \cap G = 0 \), while \( U + G / G \cong U / U \cap G \cong U \). Thus \( G \) has the index \( p^n \) in \( U + G \) and the same index in \( E \), which contains \( U + G \). Therefore \( E = U + G \), and in fact \( E = U \oplus G \), because \( U \cap G = 0 \). Similarly \( E = U \oplus H \), and \( G \cong E / U \cong H \).

Reference

1. I. Kaplansky, Infinite abelian groups, Ann Arbor, 1954.

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