A GENERALIZATION OF THE WEDDERBURN-MALCEV
THEOREM TO INFINITE DIMENSIONAL ALGEBRAS

ROBERT B. REISEL

1. Summary. In this paper we extend the Wedderburn Principal
Theorem and the Malcev Theorem for associative algebras to certain
infinite dimensional algebras. Let $A$ be an algebra over the base field
$F$ having (Jacobson) radical $N$. The Wedderburn Principal Theorem
states that if $N$ is nilpotent and if $A/N$ is a finite dimensional separable
algebra over $F$, then $A$ is cleft. The Malcev Theorem, as gen-
eralized by Tihomirov [7], states that under these hypotheses any
two cleavings of $A$ are conjugate. (See §2 for the definitions of these
terms.) Curtis [3] has generalized these theorems to the case where
$\bigcap_{k=0}^{\infty} N^k = 0$ and $A$ is complete with respect to the topology in which
the powers of $N$ form a fundamental system of neighborhoods of
zero. In §3 we show that the Wedderburn Principal Theorem holds
for such an algebra $A$ in the case where $A/N$ has countable dimension
over the base field. To generalize the Malcev Theorem, we drop this
dimensionality restriction, but the following hypothesis on the radical
is needed: for every positive integer $n$, $N^n/N^{n+1}$ is assumed to be
complete with respect to a topology in which a fundamental system of neighborhoods of zero is composed of centralizers of finite dimen-
sional separable subalgebras of $A/N$ (when $N^n/N^{n+1}$ is considered as
an $A/N$-module). With these conditions it is shown in §4 that any
two cleavings of $A$ are conjugate. The necessity for this additional
hypothesis in the Malcev Theorem is shown in §5 by an example in
which $A/N$ is of dimension $\aleph_0$ over $F$ and $N^2 = 0$.

2. Introduction. By an algebra $A$ over the base field $F$ we shall
mean an associative algebra of possibly infinite dimension over $F$.
The (Jacobson) radical of $A$ shall be denoted by $N$. To denote the
direct sum of two vector subspaces $B$ and $C$ of $A$, when $A$ is con-
sidered as a vector space over $F$, we shall use the notation $B \oplus C$. The
algebra $A$ will be called locally separable in case every finite set of
elements of $A$ is contained in a finite dimensional (over $F$) separable
subalgebra of $A$.

Presented to the Society, April 22, 1955; received by the editors March 8, 1955.
1 This paper is part of a thesis submitted by the author to the Graduate School of
Northwestern University in partial fulfillment of the requirements for the Ph.D. de-
gree. The author wishes to express his appreciation to Professor D. Zelinsky, who
directed the preparation of this paper.
If \( r \in A \) has a quasi-inverse \( r' \), then the mapping \( a \mapsto a^r = a - ar - r'a + r'ar, \) for \( a \in A \), is an automorphism of \( A \), called the quasi-inner automorphism of \( A \) generated by \( r \). (Note that \( a^r = (1 - r)^{-1}a(1 - r) \) if \( A \) has an identity element.) If \( r \circ s = r + s - rs \), then it is easy to verify that if \( r \) and \( s \) have quasi-inverses, \( (ar)^s = a^{rs} \), for \( a \in A \). If the algebra \( A \) contains a subalgebra \( S \) such that \( A = S \oplus N \), then \( A \) is said to be cleft and such an expression for \( A \) is called a cleaving of \( A \). Two cleavings of \( A \), say \( A = S \oplus N = S* \oplus N \), are said to be conjugate if \( S \) can be mapped onto \( S* \) by a quasi-inner automorphism of \( A \) generated by an element of \( N \).

A topology can be imposed on \( A \) by taking the sets \( N^k (k = 1, 2, \ldots) \) as a fundamental system of neighborhoods of zero. If we assume that the radical \( N \) has the property that \( \bigcap_{k=1}^\infty N^k = 0 \), then \( A \) with respect to this topology is a Hausdorff topological ring. In this case we shall call the topology the \( N \)-adic topology of \( A \). With the \( N \)-adic topology \( A \) is a metrizable space, so the usual idea of completeness in terms of Cauchy sequences is applicable.

3. A generalization of the Wedderburn Principal Theorem.

Lemma. Let \( A \) be an algebra over the base field \( F \) having radical \( N \) with \( \bigcap_{k=1}^\infty N^k = 0 \), such that \( A \) is complete with respect to the \( N \)-adic topology and \( A/N \) is a finite dimensional separable algebra over \( F \). Let \( E \) be a closed subalgebra of \( A \), such that \( E/(E \cap N) \) is a finite dimensional separable algebra over \( F \). Then for a cleaving \( E = D \oplus (E \cap N) \) of \( E \), there exists a cleaving of \( A \), \( A = B \oplus N \), such that \( B \supset D \).

Proof. Since \( E \) is closed, \( E \cap N \) is a (right) quasi-regular ideal of \( E \). Also \( E/(E \cap N) \) is semi-simple, so \( E \cap N \) is the radical of \( E \). Let \( H = D \oplus N \subset A = B' \oplus N \), where \( B' \oplus N \) is a cleaving of \( A \) (which exists by the result of Curtis [3] mentioned in §1). This gives another cleaving of \( H \), \( H = D' \oplus N \) with \( D' \subset B' \). \( D \cong E/(E \cap N) \) which is a finite dimensional separable algebra over \( F \). Since \( N \) is open, \( H = D \oplus N \) is an open subalgebra, hence it is closed. Thus, \( H \) is complete with respect to the induced topology. By Curtis’ results there exists a quasi-inner automorphism of \( H \) generated by an element of \( N \) that maps \( D' \) onto \( D \). This can be extended to \( A \) and it will map \( B' \) onto some \( B \) such that \( D \subset B \). This proves the lemma.

Theorem 1. Let \( A \) be an algebra over the base field \( F \) having radical \( N \) with \( \bigcap_{k=1}^\infty N^k = 0 \), such that \( A \) is complete with respect to the \( N \)-adic topology and such that \( A/N \) is a locally separable algebra of dimension \( \leq \aleph_0 \) over \( F \). Then \( A \) is cleft.

Proof. By the hypotheses there exists in \( A/N \) a nondecreasing
sequence of finite dimensional separable subalgebras $\Gamma_k$ over $F (k = 1, 2, \cdots)$ such that $A/N = \bigcup_{k=1}^\infty \Gamma_k$. Let $a_1, a_2, a_3, \cdots$ be a basis of $A/N$ over $F$ such that the first $k_n$ elements form a basis of $\Gamma_n$ over $F$ for some $n = 1, 2, \cdots$. In $A$ take representatives $a_1, a_2, a_3, \cdots$ of these and for each $n$ let $C_n$ be the closure in $A$ for the $N$-adic topology of the subalgebra generated over $F$ by $a_1, a_2, \cdots, a_{k_n}$. It is easily seen that under the canonical mapping of $A$ onto $A/N$, $C_n$ maps onto $\Gamma_n$. Also $\cap_{n=1}^{\infty} (C_n \cap N)^k = 0$, and since $C_n$ is closed, $C_n \cap N$ is the radical of $C_n$. Thus, $C_n$ is complete with respect to the induced topology and by Curtis' result there exists a subalgebra $D_n$ of $C_n$ such that $C_n = D_n \oplus (C_n \cap N)$.

Suppose we have the cleavings $C_i = D_i \oplus (C_i \cap N)$, $i = 1, 2, \cdots, n-1$, such that $D_1 \subseteq D_2 \subseteq \cdots \subseteq D_{n-1}$. Consider $C_n \subseteq C_{n-1}$. By the lemma and the above results, there exists a cleaving $C_i = D_i \oplus (C_i \cap N)$ such that $D_{n-1} \subseteq D_n$. Hence, we have a nondecreasing sequence of subalgebras of $A$, $D_1 \subseteq D_2 \subseteq D_3 \subseteq \cdots$, such that $C_n = D_n \oplus (C_n \cap N)$, $n = 1, 2, \cdots$. Take $D = \bigcup_{k=1}^\infty D_k$. Then $A = D \oplus N$ is the desired cleaving.

4. **A generalization of the Malcev-Tihomirov Theorem.** Let $A$ be an algebra over the field $F$ having radical $N$. If $M$ is any two-sided $A/N$-module, we define for $\bar{a} \in A/N$

$$M(\bar{a}) = \{ m \in M \mid m\bar{a} = \bar{a}m \}.$$ 

and for $\Gamma \subseteq A/N$,

$$M(\Gamma) = \bigcap_{\bar{a} \in \Gamma} M(\bar{a}).$$

Note that $M(\bar{a})$ and $N(\Gamma)$ are additive subgroups of $M$.

**Theorem 2.** Let $A$ be an algebra over the base field $F$ having radical $N$ with $\bigcap_{k=1}^\infty N_k = 0$, such that $A$ is complete with respect to the $N$-adic topology and such that $A/N$ is locally separable over $F$. For every positive integer $n$, $N_n = N^n/N^{n+1}$ is assumed to be complete with respect to the topology $\mathfrak{a}_n$ in which the sets $N_n(\Gamma)$, for $\Gamma$ running through all finite dimensional separable subalgebras of $A/N$, form a fundamental system of neighborhoods of zero. Then any two cleavings of $A$ are conjugate.

**Remark.** It is easily seen that the sets $N_n(\Gamma)$, for $\Gamma$ running through all finite dimensional separable subalgebras of $A/N$, form a fundamental system of neighborhoods of zero for a topology on $N_n$. However, it is not, in general, a Hausdorff topology. We take completeness in the sense of Bourbaki [2, Chap. 2, §3] and we shall use the Bourbaki terminology throughout the proof.
Proof. Suppose the cleavings of $A$ are $A = S \oplus N = S^* \oplus N$. If $a \in S$, by $a^*$ we mean that element of $S^*$ such that $a \equiv a^* \pmod{N}$.

Case 1. Assume that $N^2 = 0$. In this case the quasi-inner automorphism of $A$ generated by $r \in N$ takes the form $a^r = a - ar + ra$, $a \in A$. Considering $N$ as an $A/N$-module, we have that two elements of $N$ generate quasi-inner automorphisms of $A$ mapping $a \in A$ onto the same element if and only if their difference lies in the $N$-module $N(a)$.

Let $C$ be a finite dimensional separable subalgebra of $S$ with $\Gamma \subseteq A/N$ its image under the canonical mapping of $A$ onto $A/N$. Then there exists a finite dimensional separable subalgebra $C^*$ of $S^*$, such that $C \oplus N = C^* \oplus N$. By the Malcev-Tihomirov Theorem [7], there exists a quasi-inner automorphism of $C \oplus N$ generated by an element $r \in N$ such that $C^r = C^*$. The set of all $r \in N$ generating this quasi-inner automorphism we shall call $R_T$. By the above results, $R_T$ is just a coset of $N(\Gamma)$ in $N$.

We next prove that the collection $\{R_T\}$, where $\Gamma$ runs through all finite dimensional separable subalgebras of $A/N$, forms a filter base. Let $\Gamma$ and $\Gamma'$ be two finite dimensional separable subalgebras of $A/N$. By the local separability of $A/N$, there exists a finite dimensional separable subalgebra $\Delta$ of $A/N$ containing $\Gamma$ and $\Gamma'$. Then clearly $R_\Delta \subseteq R_T \cap R_{T'}$, so $\{R_T\}$ is a filter base. When we consider $N$ with the topology $\mathfrak{3}$ formed by taking the sets $N(\Gamma)$ as neighborhoods of zero, we see that $\{R_T\}$ forms a Cauchy filter base because $R_T$ is a coset of $N(\Gamma)$. By the completeness of $N$, this filter base has a limit $r \in N$. Therefore, $r$ is in the closure of each $R_T$; and since $R_T$, being an open coset, is closed, $r \in \cap R_T$. Then $S^r \subseteq S^*$. Since the mapping is clearly onto, we have $S^r = S^*$, which concludes the proof of Case 1.

Remark. The completeness hypothesis on $N$ is not unduly strong because convergence of all Cauchy filters is equivalent to convergence of Cauchy filters with a base of cosets of the $N(\Gamma)$'s. Indeed, let $\mathfrak{F}$ be any Cauchy filter in $N$ for the topology $\mathfrak{3}$. For each $N(\Gamma)$, $\mathfrak{F}$ contains exactly one coset of $N(\Gamma)$; let $\mathfrak{F}'$ be the collection of these cosets. Then $\mathfrak{F}'$ is a Cauchy filter base, and being coarser than $\mathfrak{F}$, its convergence implies that of $\mathfrak{F}$. On the other hand, if $\mathfrak{F}$ converges to $r \in N$, $r$ is a limit point of $\mathfrak{F}$, hence of $\mathfrak{F}'$, so it is a limit of $\mathfrak{F}'$. Thus, $\mathfrak{F}$ converges if and only if $\mathfrak{F}'$ converges.

Case 2. We now remove the restriction that $N^2 = 0$. We use induction. Suppose there exist elements $r_i \in N^i$, $i = 1, 2, \ldots, n - 1$, such that

\[(1) \quad a^{r_1 \cdots r_{n-1}} \equiv a^* \pmod{N^*}, \quad \text{for all } a \in S.\]
Let \( S' \) be the image of \( S \) under the quasi-inner automorphism of \( A \) generated by \( r_1 \circ \cdots \circ r_{n-1} \). Then \( S' \oplus N^n = S^* \oplus N^n \). Let \( \psi_n \) be the canonical mapping of \( A \) onto \( A/N^{n+1} \). (We shall write the mapping symbol on the right.) We define \( N_n = N^n \psi_n = N^n/N^{n+1} \) and \( A_n = S' \psi_n \oplus N_n = S^* \psi_n \oplus N_n \). Then \( N_n^2 = 0 \) and \( A_n/N_n \) is semi-simple and locally separable. Therefore, \( N_n \) is the radical of \( A_n \) and we can apply the process of Case 1 to the algebra \( A_n \). We thus have a collection \( \{ R_{n,1} \} \) of cosets of \( N_n(\Gamma) \) in \( N_n \), where \( \Gamma \) runs through all finite dimensional separable subalgebras of \( A/N \). This collection is a Cauchy filter base for the topology \( \mathcal{T}_n \). By the hypothesis of completion of \( N_n \) with respect to this topology, there exists an element \( r_n \in N^n \) such that \( r_n \psi_n \in \bigcap R_{n,1} \). Then \( r_n \psi_n \) generates a quasi-inner automorphism of \( A_n \) that maps \( S' \psi_n \) onto \( S^* \psi_n \), that is, for this \( r_n \in N^n \) the relation (1) is true with \( n \) replaced by \( n + 1 \). This completes the induction step.

We have thus constructed a sequence \( \{ r_n \}, r_n \in N^n (n=1, 2, \ldots) \) such that (1) holds for each positive integer \( n \). Hence, with respect to the \( N \)-adic topology, the sequence \( \{ a^o \circ \cdots \circ r_n - a^* \} \) has limit 0. The sequence \( r_1, r_1 \circ r_2, r_1 \circ r_2 \circ r_3, \ldots \) is a Cauchy sequence in \( A \) for the \( N \)-adic topology and hence has a limit \( r \in N \). Also the sequence obtained from this one by taking quasi-inverses of each term is a Cauchy sequence and converges to \( r' \in N \). For each \( a \in S \) we have \( a^* - a^o \circ \cdots \circ r_n = a^* - a + a(r_1 \circ \cdots \circ r_n) + (r_1 \circ \cdots \circ r_n)'a - (r_1 \circ \cdots \circ r_n)'a(r_1 \circ \cdots \circ r_n) \). Taking limits of both sides with respect to the \( N \)-adic topology of \( A \), we get \( a^* = a' \). Since this is true for each \( a \in S \), we have \( S' \subseteq S^* \) and clearly this mapping is onto. Thus, \( r \in N \) generates a quasi-inner automorphism of \( A \) which maps \( S \) onto \( S^* \). This proves Theorem 2.

**Corollary 1.** Let \( A \) be an algebra over the base field \( F \) having radical \( N \) with \( \bigcap_{n=1}^\infty N^n = 0 \), such that \( A \) is complete with respect to the \( N \)-adic topology and such that \( A/N \) is a finite dimensional separable algebra over \( F \). Then any two cleavings of \( A \) are conjugate.

**Proof.** For any positive integer \( n \), all the \( N_n(\Gamma) \) can be deleted except \( N_n(A/N) \) itself and the topology \( \mathcal{T}_n \) is unchanged. A Cauchy filter on \( N_n \) for the topology \( \mathcal{T}_n \) contains a coset of \( N_n(A/N) \) and any element of this coset is a limit of the filter. Hence, \( N_n \) is complete and the result follows from Theorem 2.

Corollary 1 is Curtis' generalization of the Malcev Theorem [3]. Corollary 2, which follows below, is a result of Kuročkin [5].

**Corollary 2.** Let \( A \) be an algebra over the base field \( F \) having a finite dimensional radical \( N \) such that \( A/N \) is locally separable. Then any two cleavings of \( A \) are conjugate.
Proof. Since $N$ is finite dimensional, it is nilpotent, i.e., there exists a positive integer $m$ such that $N^m = 0$. From this it easily follows that $A$ is complete with respect to the $N$-adic topology. Let $n$ be a positive integer smaller than $m$. If $\Gamma$ and $\Gamma'$ are finite dimensional separable subalgebras of $A/N$ such that $\Gamma \supset \Gamma'$, then $N_n(\Gamma) \subset N_n(\Gamma')$. Since $N_n$ is finite dimensional and $A/N$ is locally separable, there exists a finite dimensional separable subalgebra $\Delta$ of $A/N$ such that $N_n(\Delta)$ has minimal dimension in the set of all $N_n(\Gamma)$. Although $\Delta$ is not unique, it is easily seen that all such minimal $N_n(\Delta)$ are equal. Thus a fundamental system of neighborhoods of zero consists just of the set $N_n(\Delta)$. As in the proof of Corollary 1, it follows that $N_n$ is complete with respect to the topology $3_n$. The conclusion of the corollary now follows from the theorem.

5. Example. We shall construct an algebra $A$ over the field $F$ of rational numbers having radical $N$ with $N^2 = 0$, such that $A/N$ is a separable field of dimension $\aleph_0$ over $F$, but such that $A$ has two nonconjugate cleavings. This will show the necessity of having some additional hypothesis on the radical, as in Theorem 2, in order that the Malcev Theorem be true when $A/N$ is not finite dimensional. Note that since $N^2 = 0$, $A$ will be complete with respect to the $N$-adic topology.

Let $\zeta_n$ be a primitive $2^{n+1}$th root of unity for $n = 0, 1, 2, \cdots$, and $\zeta_n = 1$ for $n < 0$. For each nonnegative integer $n$, we consider the field $K_n = F(\zeta_n)$ and let $K = \bigcup_{n=0}^{\infty} K_n$. $K$ is then a separable field of dimension $\aleph_0$ over $F$. Let $N$ be the set of all $N_0$-tuples over $K$ in which there are only a finite number of nonzero terms. With the usual operations $N$ is a vector space over $K$; let $e_n$ be the $\aleph_0$-tuple with 1 in the $n$th place and zeros elsewhere. We define $A$ to be the direct sum of the $F$-spaces $K$ and $N$. To make $A$ into an algebra, we define multiplication as follows:

1. For elements in $K$, multiplication is the field multiplication.
2. For elements in $N$, we define $N^2 = 0$.
3. For $k \in K$ and $r = \sum k_n e_n \in N$ (where the $k_n$ are in $K$ and all but a finite number are zero), we define $kr = \sum (kk_n)e_n$.
4. For each $j = 1, 2, \cdots$, let $\sigma_j$ be the $F$-automorphism of the field $K$ such that $\sigma_j(\zeta_n) = \zeta_n^{2^j+1}$, for each $n$. Then for every $k \in K$ we define $e_j k = \sigma_j(k)e_j$.
5. Lastly if $r = \sum k_j e_j \in N$, we define $rk = \sum k_j \sigma_j(k)e_j$.

It can be shown that $A$ is then an algebra over $F$ having an identity element, namely the identity element of $K$, and that $N$ is the radical of $A$. Also $A/N$ is isomorphic to $K$ and $A = K \oplus N$ is a cleaving of $A$. 
We shall now give another cleaving of $A$. For each non-negative integer $n$, we define $r_n = \sum j (\xi_j e_j - e_{j, n})$, which is an element of $N$. By an obvious induction on $q$, we find that $(\xi_n - r_n)^q = \xi_n^q - \sum j (\xi_j^q e_j - e_{j, qn})$, from which it follows that $\xi_n - r_n$ is a primitive $2^{n+1}$th root of unity. For each $n$ we consider the field $K_n^* = F(\xi_1 - r_1, \xi_2 - r_2, \cdots, \xi_n - r_n)$ and let $K^* = \bigcup_{n=0}^\infty K_n^*$, where $K_0^* = F$. Then for each $n$, $K_n \oplus N = K_n^* \oplus N$, and $A = K^* \oplus N$.

Now let $s \in N$ and consider $(1 + s)\xi_n(1 - s)$. This is a (primitive) $2^{n+1}$th root of unity. Hence if it lies in $K^*$, it must be a power, say the $m$th, of $\xi_n - r_n$, that is, $(\xi_n - r_n)^m = \xi_n - (\xi_n s - s\xi_n)$. Taking cosets mod $N$, we have that $\xi_n^m = \xi_n$, hence also $(\xi_n - r_n)^m = \xi_n - r_n$. Thus we have $\xi_n s - s\xi_n = r_n$. If we write $s = \sum k_j e_j$, with $k_j \in K$, we have from this that $\xi_n k_j - k_j \sigma_j (\xi_n) = \xi_n - \sigma_j (\xi_n)$, $j = 1, 2, \cdots$, that is, $\xi_n k_j (1 - \xi_n^{2j}) = \xi_n (1 - \xi_n^{2j})$. If $j \leq n$, $1 - \xi_n^{2j} \neq 0$, so $k_j = 1$. However, since $N$ consists of the $N_n$-tuples with only a finite number of nonzero terms, this shows that if $s \in N$, the isomorphism $k \rightarrow (1 + s)k(1 - s)$ cannot map every $\xi_n$ into $K^*$. Hence, $A = K \oplus N = K^* \oplus N$ are two nonconjugate cleavings of $A$.

References