ON A PROPERTY OF MONOTONE AND CONVEX FUNCTIONS

G. SZÉKERES

We shall deal with real functions which have continuous second derivatives in some open interval \((a, b)\), \(-\infty < a < b < \infty\). The interval of definition of \(f(x)\) is denoted by \(I(f)\). \(\phi(x)\) is called convex (from below) if \(\phi''(x) \geq 0\), concave if \(\phi''(x) \leq 0\) in \(I(\phi)\).

If \(\phi(x)\) is monotone increasing and convex, and \(\psi(x)\) is monotone increasing and concave such that the range of \(\phi(x)\) is contained in \(I(\psi)\), then

\[
(1) \quad f(x) = \psi(\phi(x))
\]

is also monotone increasing, but usually neither convex nor concave. The question arises, under what conditions can \(f(x)\) be represented in the form (1).

**Theorem 1.** If \(f(x)\) is strictly monotone increasing and has a continuous second derivative in \(I(f)\) then it has a representation (1).

Theorem 1 states that there is a strictly increasing concave function \(\psi(u)\) with continuous second derivative such that \(\psi_1(u) = f(\psi(u))\) is concave. This is equivalent to

\[
(2) \quad \psi_1'(u) = f''(\psi(u)) [\psi'(u)]^2 + f'(\psi(u)) \psi''(u) \leq 0,
\]

or if \(\phi(x)\) is the inverse of \(\psi(u)\), to

\[
(3) \quad f''(x)/f'(x) \leq \phi''(x)/\phi'(x).
\]

Let \(f''(x)\) denote \(f''(x)\) if \(f''(x) \geq 0\) and 0 if \(f''(x) < 0\). Consider the function

\[
(4) \quad \phi_d(x) = \int_d^x e^{p(y)} dy
\]

where \(d\) is any fixed number in \((a, b)\) and

\[
(5) \quad p(y) = \int_d^y [f''(t)/f'(t)] dt.
\]

Clearly \(\phi_d(x) > 0\) and

Received by the editors February 21, 1955 and, in revised form, May 26, 1955.

351
\( \frac{\phi''(x)}{\phi'(x)} = \frac{f''(x)}{f'(x)} \)

so that (3) is satisfied, also

(7) \[ \phi''(x) \geq 0. \]

This proves the theorem.\(^1\)

If \( f(x) \) is bounded and \( I(f) \) is finite, the question comes up whether \( \phi(x) \) itself can be chosen to be bounded. This is answered by

**Theorem 2.** If \( f(x) \) is bounded, strictly increasing and has a continuous second derivative in \((a, b)\), then it can be represented in the form (1) with bounded \( \phi(x) \) if and only if the integral

(8) \[ \int_a^b e^p(y) \, dy \]

converges, where \( a < d < b \) and \( p(y) \) is the function defined under (5).

We shall see presently that boundedness of \( f(x) \) does not necessarily imply finiteness of (8).

To prove Theorem 2 we first note that, by (4), \( \phi_0(x) \) is bounded from above if (8) is finite, and also bounded from below if \( f(x) \) is bounded since

\[
\phi_0(x) \equiv \int_d^y \left\{ \exp \int_d^y \frac{f''(t)}{f'(t)} \, dt \right\} \, dy
= \left[ f(x) - f(d) \right] / f'(d)
\]

by (4) and (5).

Suppose now that (8) diverges, so that \( \phi_0(x) \) is unbounded from above, and let \( \phi_1(x) \) be any function which has the properties (3) and (7). By taking a suitable linear combination \( \phi(x) = c_0 \phi_0(x) + c_1 \) we can achieve that \( \phi(d) = 0, \phi'(d) = 1 \). Now from (3) and (6),

\[
\frac{d}{dx} \log \phi'(x) \geq \frac{f''(x)}{f'(x)} \geq \frac{d}{dx} \log \phi_0'(x)
\]

which implies

\[ \phi'(x) \geq \phi_0'(x), \quad \phi(x) \geq \phi_0(x) \quad \text{for } x > d. \]

This shows that \( \phi_0(x) \) is in a sense the “least convex” among all possible solutions and that \( \phi(x) \), hence also \( \phi_1(x) \), is unbounded.

---

\(^1\) I am indebted to G. Lorentz for a substantial shortening of the original argument. His proof, which is reproduced above, contributed greatly to a simplified treatment of another part of the paper.
Theorems 1 and 2 have obvious dual formulations.

**Theorem 1.** Under the same conditions as in Theorem 1, \( f(x) \) can be represented in the form

\[
(1^*) \quad f(x) = \phi_1(\psi(x))
\]

where \( \phi_1 \) is convex and \( \psi \) concave.

**Theorem 2.** If \( f(x) \) is as in Theorem 2, then it can be represented in the form \((1^*)\) with bounded \( \psi(x) \) if and only if

\[
(8^*) \quad \int_a^d e^{q(y)} \, dy
\]

is finite, where

\[
(5^*) \quad q(y) = \int_y^d \left[ f''(t)/f'(t) \right] dt.
\]

Here \( f''(t) \) denotes \(-f''(t)\) if \( f''(t) \leq 0 \) and \( 0 \) if \( f''(t) > 0 \).

The following example shows that boundedness of \( f(x) \) does not necessarily imply finiteness of \((8)\) or \((8^*)\). Take \( f(x) = 2x + x^3 \sin (1/x) \) over the interval \((0, 1)\). It is easily seen that \( f''(x) \) has a zero \( x_n \) between \( 2/(2n+1)\pi \) and \( 2/(2n-1)\pi \), and

\[
f''(x) \begin{cases} > 0 & \text{for } x_{2m} < x < x_{2m-1}, \\
< 0 & \text{for } x_{2m+1} < x < x_{2m}, \quad m = 1, 2, \ldots.
\end{cases}
\]

It can also be shown easily that

\[
x_n = 1/n\pi + 2/n^3\pi^3 + O(n^{-6}),
\]

\[
f'(x_n) = 2 - (-1)^n + O(n^{-2}),
\]

so that

\[
\int_{x_{2m}}^{x_{2m+1}} \left[ f''(t)/f'(t) \right] dt = - \log 3 + O(m^{-2})
\]

and

\[
q(y) = m \log 3 + O(1) \quad \text{for } x_{2m+1} < x < x_{2m-1}.
\]

This shows that \( \int_0^1 e^{q(y)} \, dy \) is divergent.

University of Adelaide