ON SEMI-GROUPS OF UNBOUNDED NORMAL OPERATORS

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In this note we discuss the integral representation of a semi-group of unbounded normal operators. The result for a semi-group of bounded normal operators can be found in Sz. Nagy [4]. Recently A. Devinatz [1] obtained a similar result for semi-groups of unbounded self-adjoint operators. The following theorem is proved.

**Theorem.** Let \( \{N_t; t>0\} \) be a semi-group (i.e., \( N_t N_s = N_{t+s} \)) of normal operators on a Hilbert space \( \mathcal{H} \). Let \( D_t \) be the domain of \( N_t \) (each \( D_t \) dense in \( \mathcal{H} \)) and let \( D = \bigcap_{t>0} D_t \) then we suppose that \( N_t x \) is weakly continuous as a function of \( t (t>0) \) for each fixed \( x \in D \). Then there exists a unique complex spectral resolution \( K(\lambda) \) whose support is contained in \( \lambda_1 \geq 0, \lambda = \lambda_1 + i\lambda_2, \) such that

\[
N_t = \int_{\lambda_1 \geq 0} \lambda_1 e^{i\lambda_2 t} K(d\lambda), \quad t > 0.
\]

**Proof.** First recall that if \( N \) is a normal operator then \( N = AU = UA \) where \( U \) is unitary and \( A \) is self-adjoint and has the same domain as \( N \). In fact if \( K(\lambda) \) is the spectral resolution of \( N \) then we can define \( A \) and \( U \) as follows,

\[
(2) \quad A = \int |\lambda| K(d\lambda),
\]

\[
(3) \quad U = \int s(\lambda) K(d\lambda) \text{ where } s(\lambda) = \begin{cases} \lambda/|\lambda|, & \lambda \neq 0, \\ 1, & \lambda = 0. \end{cases}
\]

Let us also note that if \( p \) is an integer then \( (N^*)^p = (A^p U^*)^* = A^p U^{-p} \) and that \( (N^*)^p = (A U^*)^p = A^p U^{-p} \), hence \( (N^*)^p = (N^*)^p \).

We now prove a series of simple statements which taken together yield our theorem.

(a) If \( t \) and \( s \) are commensurable then \( N_t N_s^* = N_s^* N_t \). This is a trivial consequence of the semi-group property of \( \{N_t, t>0\} \) and the fact that \( (N_t^*)^n = (N_t^*)^n \).

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387
(b) If $x \in D$ then $N_t N^*_s x = N^*_s N_t x$ for all $t, s > 0$. First if $x \in D$ then $N_t x \in D$ and $N^*_s x \in D$ for all $t > 0$, since if $x \in D$ then $N_{t+s} x$ exists and $N_s N_t x = N_{t+s} x$. Thus $N_t x \in D_s$ for all $s$ and hence is in $D$. To show $N^*_t x \in D$, let $t_0 > 0$ be fixed, then $N_t x \in D$ for all $s > 0$ as seen above, hence $N^*_t N_x x$ exists. (Domain of $N^*_t$ is $D_t$ since $N_t$ is normal.) Let $s = nt_0$, $n = 1, 2, \ldots$, then since $s$ and $t_0$ are commensurable $N^*_t N_s x = N_s (N^*_t x)$ or $N^*_t x \in D_{nt_0}$ for $n = 1, 2, \ldots$. But by the semi-group property of $N_t$, $t > 0$, the $D_t$'s are a decreasing collection of sets and therefore $D = \cap_{t > 0} D_t = \lim_{t \to \infty} D_t = \lim_{n \to \infty} D_{nt_0}$ since $t_0 > 0$. Thus $N^*_t x \in D$.

If $t, s$ are given and $x \in D$ then $N_t N^*_s x$ and $N^*_s N_t x$ both exist. Let $t_n \to t$, $t_n$ and $s$ commensurable. A standard argument making use of the continuity property of $\{N_t, t > 0\}$ shows that for a fixed $x \in D$ $(N^*_s N_t x, y) = (N_t N^*_s x, y)$ for all $y \in D_s$, but $D_s$ is dense in $\mathfrak{X}$ and hence $N^*_s N_t x = N_t N^*_s x$.

(c) If we define $A_t = N^*_t N_{t/2}$, $t > 0$, then $\{A_t, t > 0\}$ is a semi-group of self-adjoint operators such that

$$A_t = A^*_t = \int_0^\infty \lambda dE(\lambda),$$

$t > 0$.

It is clear from the definition that each $A_t$ is a positive self-adjoint operator and moreover $A_t^2 = N^*_t N_t$. Since a positive self-adjoint operator has a unique positive self-adjoint square root it follows that $A_t$ is the operator defined by (2) for $N_t$. Thus $D_{A_t} = D_t$ and $D = \cap_{t > 0} D_{A_t}$.

If $x \in D$ then $A_t A_s x = N^*_t N^*_s N_{t/2} N_{s/2} x$ and this is defined since $N_t D \subseteq D$ and $N^*_s D \subseteq D$ for all $t > 0$. Moreover by (b)

$$A_t A_s x = N^*_t N^*_s N_{t/2} N_{s/2} x = N^*_s N_{(t+s)/2} x = A_{t+s} x,$$

since in general $N^*_{(t+s)/2} \supseteq N^*_t N^*_s$. Also

$$A_t A_s x = N^*_s N^*_t N_{t/2} N_{s/2} x = N^*_s N_{s/2} N^*_t N_{t/2} x = A_s A_t x.$$

Consequently if $x \in D$, $A_t A_s x = A_s A_t x = A_{t+s} x$. Now the same argument as used by Sz. Nagy [4, p. 74] shows that $(A_s x, x)$ for each fixed $x \in D$ is bounded above as a function of $t$ in every interval $0 < a \leq t \leq b$. This implies, Sz. Nagy [4, p. 73], that $(A_s x, x)$ is continuous for $t > 0$ and $x \in D$. We would like to apply Devinatz' theorem at this point to the $A_t$'s but we do not know a priori that the $A_t$'s form a semi-group. The following argument is almost word for word that of Devinatz [1, p. 102]. Define $H_t = A^*_t$. Clearly $\{H_t, t > 0\}$
is a semi-group. In addition using (a) we see that $H_n = A_1^n = (N_{1/2}^n N_{1/2})^n = N_{n/2}^n N_{n/2} = A_n$. Furthermore the uniqueness of square roots of positive self-adjoint operators implies that for any integers $n, m, H_{n/2}^m = A_{n/2}^m$. Now, there exists a countable set of mutually orthogonal manifolds $\{M_k\}$, whose direct sum is the whole space and such that, for all $t > 0$, $H_t = \sum_{k=1}^n \oplus H_t^{(k)}$, where $H_t^{(k)}$ is a bounded self-adjoint operator on $M_k$ and is the restriction of $H_t$ to $M_k$ (Sz. Nagy [4, p. 49]). That is, $x \in D_n$ if and only if $\sum_{k=1}^n \|H_t^{(k)}x_k\|^2 < \infty$, where $x_k \in M_k$, and $x = \sum_{k=1}^n x_k$. Then $H_x = \sum_{k=1}^n H_t^{(k)}x_k$.

Given any $t > 0$ there exists an $m/2^n \geq t$. From the semi-group property of $\{N_t, t > 0\}$ we know that $D_{m/2^n} \subset D_t$ ($D_t$ is also the domain of $A_t$). Thus for every $x_k \in M_k$, $x_k \in D_{m/2^n} \subset D_t$. Consequently, since $H_m = H_m^n$ and by the continuity of $(A_t x_k, x_k)$ and $(H_t x_k, x_k)$ as functions of $t$, we must have $A_t x_k = H_t^{(k)} x_k = H_t x_k$. This implies that $H_t = A_t$ (Sz. Nagy [4, p. 35]), and hence (4) is proved.

For any $t > 0$ the above argument shows that $M_k \subset D_t$ for $k = 1, 2, \cdots$, and hence $M_k \subset D$ for all $k$. Thus if $x \in \mathcal{E}$ and $x = \sum_{k=1}^n x_k$ then $y_n = \sum_{k=1}^n x_k$ is in $D$ and $y_n \to x$. That is $D$ is dense in $\mathcal{E}$. Moreover for any $t > 0$ if $x \in D_t$ then $A_t y_n = \sum_{k=1}^n A_t x_k = \sum_{k=1}^n H_t^{(k)} x_k = H_t x$. Thus for any $x \in D_t$ there exists a sequence $y_n \in D$ such that $y_n \to x$ and $A_t y_n \to A_t x$.

For each $t > 0$ let $U_t$ be the unitary operator defined by (3) such that $N_t = A_t U_t = U_t A_t$. From (2) and (4) it follows that $A_t$ and $N_t$ have the same null space, $\mathfrak{N}$, for all $t, s > 0$. $\mathfrak{N}$ is a closed linear manifold since the operators in question are closed. If we write $\mathfrak{H} = \mathfrak{N} \oplus \mathfrak{N}$ where $\mathfrak{N}$ is the orthogonal complement of $\mathfrak{N}$, then $\mathfrak{N}$ can be characterized as either the closure of $R_{A_t}$ or the closure of $R_{N_t}$ for any $t > 0$. (If $T$ is an operator $R_T$ denotes the range of $T$.) Thus $\mathfrak{N}$ is the null space of $A_t$, $N_t$, and $N_t^*$ for all $t > 0$ and if we write (3) with the proper subscript we see that $\mathfrak{N}$ is also the null space of $U_t$ for all $t > 0$. (Note that $K_t(\{0\})$ is the projection on $\mathfrak{N}$.) It is now clear that all of the above operators are reduced by $\mathfrak{N}$. (A normal operator is always reduced by its null space.) Thus we can restrict all the operators in question to $\mathfrak{N}$. We assume $\mathfrak{N} \neq \{0\}$ since in this case everything is trivial.

(d) If $\bar{D} = D \cap \mathfrak{N}$ then $\bar{D}$ is dense in $\mathfrak{N}$. Assume $\bar{D}$ not dense in $\mathfrak{N}$ then there exists $r \in \mathfrak{N}, r \neq 0$, such that $r \perp \bar{D}$. Let $x \in D$, then $x = x_k + n$, $x_k \in \mathfrak{N}, n \in \mathfrak{N}$. Since $n \in D$ and $D$ is linear we see that $x_k = x - n \in D$ and hence in $\bar{D}$. Therefore $(r, x) = (r, x_k) + (r, n) = 0$ which implies that $r = 0$ as an element of $\mathfrak{N}$ since $D$ is dense in $\mathfrak{N}$. But this contradicts the fact that $r \neq 0$ as an element of $\mathfrak{N}$.

(e) In this section all operators are considered as operators on $\mathfrak{N}$. 

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If we define $U_0 = I$ and $U_{-t} = U_t^{-1}$ then \{ $U_t; -\infty < t < \infty$ \} is a strongly continuous group of unitary operators on $\mathfrak{A}$.

First we show that $A_t U_t x = U_t A_t x$ for $x \in \mathfrak{B}$. Note that $x \in \mathfrak{B}$ implies $A_t x \in \mathfrak{B}$ by the same argument as that used in (b), and also that (b) and the definition of $A_t$ implies $A_t N_t x = N_t A_t x$ for all $x \in \mathfrak{B}$. Thus for all $x \in \mathfrak{B}$ we have

$$A_t N_t x = N_t A_t x, \quad A_t A_s U_t x = A_s U_t A_t x.$$ 

However, the semi-group property of \{ $A_t, t > 0$ \} implies $A_t A_s = A_s A_t$, hence

$$A_s A_t U_t x = A_t A_s U_t x$$

and from this we conclude that $A_t U_t x = U_t A_t x$ since $A_t x$ is one-to-one in $\mathfrak{A}$, i.e., $A_t^{-1}$ exists. A consequence of this is that $U_t x \in \mathfrak{B}$ if $x \in \mathfrak{B}$. Therefore for $x \in \mathfrak{B}$, $N_t N_t x = N_{t+t} x = A_{t+t} U_{t+t} x = A_t A_s U_t U_s x$ or $U_{t+t} x = A_{t+t} A_{t+t} U_t U_s x = U_{t+t} A_t U_t x$. But the $U_t$'s are bounded and $\mathfrak{B}$ is dense in $\mathfrak{A}$ thus $U_t U_s = U_{t+s} = U_{t+s}$, and if we define $U_0 = I$ and $U_{-t} = U_t^*$ it is clear that \{ $U_t; -\infty < t < \infty$ \} is a group of unitary operators on $\mathfrak{A}$.

We now investigate the continuity properties of this group. To this end we first note that an immediate consequence of (4) is that $A_t$ is strongly continuous on $\mathfrak{B}$ even at $t = 0$, and that $A_t x$ is strongly left continuous at $t = t_0$ for $x \in \mathfrak{B}$, $t_0 = D = D_0 \cap \partial T$ (if $s \leq t$, then $D_t \subset D_s$ and hence $D_t \subset D_s$). Suppose $t_0 > 0$ and $t_n \uparrow t_0$, $0 < t_n < t_0$, then $D_t \subset D_{t_n}$. Let $x \in R_{t_0}$, $y \in \mathfrak{B}$ then $x = A_t z$ where $z \in D_{t_0} \subset D_s$. Thus we have

$$| (U_{t_n} x_0, y) - (U_{t_0} x_0, y) | = | (U_{t_0} A_{t_0} z, y) - (U_{t_0} A_{t_0} z, y) | \leq | (U_{t_0} - A_{t_0}) z, y | + | (U_{t_0} A_{t_0} - U_{t_0} A_{t_0}) z, y | \leq \| (A_t - A_{t_0}) z \| \| y \| + \| (A_t - A_{t_0}) z \| \| y \| + \| (z, N_{t_0} y) - (z, N_{t_0} y) | \rightarrow 0 \text{ as } t_n \uparrow t_0.$$ 

Since $R_{t_0}$ and $\mathfrak{B}$ are dense in $\mathfrak{A}$ and $\| U_t \| = 1$ it follows that \{ $U_t, t < 0$ \} is weakly right continuous. However, weak right continuity at any one point $t_0$ implies weak right continuity for all $t$ since $([U_{t+h} - U_t] x, y) = ([U_{t+h} - U_t] x, U_{t+h} - U_t y)$. A minor modification in the proof of Theorem 9.2.2 in Hille [3] shows that $U_t$ is strongly continuous for all $t$. This completes the proof of (e).

From (3) we see that $U_t$ is the identity on $\mathfrak{N}$ and thus \{ $U_t; -\infty < t < \infty$ \} is a strongly continuous group of unitary operators on all of $\mathfrak{N}$. (We no longer restrict the operators in question to $\mathfrak{A}$.) The spectral theorem for unitary groups guarantees the existence.
of unique spectral resolution $F(\lambda_2)$ such that

$$U_t = \int_{-\infty}^{\infty} e^{it\lambda} dF(\lambda), \quad -\infty < t < \infty.$$  

In (e) we saw that $A_tU_t x = U_t A_t x$ for all $x \in D$. But for any $x \in D$ we have $x = n_t + r$ where $n \in \mathfrak{N}$ and $r \in D$, hence $A_t U_t x = U_t A_t x$ for all $x \in D$ since $\mathfrak{N}$ is the common null space of these operators. For any $x \in D_1$ there exists $y_n \in D$ such that $y_n \to x$ and $A_1 y_n \to A_1 x$. See (c).

Hence

$$U_t A_1 x = U_t \left[ \lim_{n \to \infty} A_1 y_n \right] = \lim_{n \to \infty} U_t A_1 y_n = \lim_{n \to \infty} A_1 U_t y_n.$$  

Moreover $U_t y_n \to U_t x$ and since $A_1$ is closed we have $U_t A_1 x = A_1 U_t x$. Thus $U_t A_1 \subset A_1 U_t$ for $-\infty < t < \infty$. Therefore by Fuglede's theorem [2] we obtain $E(\lambda_t) U_t = U_t E(\lambda_t)$ for all $t$ and $\lambda_t$ and then finally that $E(\lambda_t) F(\lambda_2) = F(\lambda_2) E(\lambda_t)$ for $\lambda_1$ and $\lambda_2$.

Putting $K(d\lambda) = E(d\lambda_1) F(d\lambda_2)$ we have

$$N_t = A_1 U_t = \int_{\lambda_t \in \mathfrak{N}} \int_{\lambda_1 \geq 2} \lambda_1^t d\lambda_1 d\lambda_2 K(d\lambda), \quad t > 0.$$  

Clearly $K(A)$ is unique on $\mathfrak{N}$ but since $\mathfrak{N} = K(\{0\}) \mathfrak{C}$ it follows that $K(A)$ is unique on all of $\mathfrak{C}$.

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References


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