ON THE CONVERGENCE OF FOURIER SERIES OF FUNCTIONS IN AN L^p CLASS

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1. In this paper, we shall present theorems concerning the convergence of certain subsequences of the full sequence of partial sums of the Fourier series of functions which belong to some L^p class, $1 < p \leqq 2$. To a large extent, our theorems are based on the following well-known theorem of Kolmogoroff [1, p. 251]: *if $\{n_k\}$ is a lacunary sequence of integers, and if $f(x)$ is a function of class L^2, then the subsequence $s_{n_k}(x; f)$ of partial sums of the Fourier series of $f(x)$ converges to $f(x)$ almost everywhere.* By lacunary sequence, we mean, of course, that there is a $\lambda > 1$ such that $n_{k+1}/n_k \geqq \lambda$ for all k. In our theorems, we are able to prove almost everywhere convergence for considerably larger subsequences, although we lose some precision in locating the indices.

2. We let the series

$$\sum_{n=-\infty}^{+\infty} c_n e^{inx} \tag{1}$$

be the Fourier series of the function $f(x)$ and consider first, for reasons of simplicity, the case when $f(x)$ belongs to the class L^2. As a matter of notation, we shall let $[y]$ denote the greatest integer less than or equal to y and make the following definition:

$$L_k = \left[\frac{n_{k+1} - n_k}{\log n_{k+1}}\right].$$

We may now state our first theorem.

THEOREM 1. *If $f(x)$ belongs to L^2, and $\{n_k\}$ is lacunary, then there is a sequence of positive integers $\{m_\nu\}$ containing L_k consecutive terms in each interval (n_k, n_{k+1}) such that the subsequence*

$$s_{m_\nu}(x; f), \qquad \nu = 1, 2, \cdots,$$

of partial sums of the Fourier series of $f(x)$ converges almost everywhere to $f(x)$.

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We introduce the following notation:

$$\epsilon_k = \sum_{|n|=n_k+1}^{n_{k+1}} |c_n|^2,$$

$$\delta_k^{(\mu)} = (\log n_{k+1}) \sum_{|n|=n_k+\mu L_k+1}^{n_k+(\mu+1)L_k} |c_n|^2, \quad \mu = 0, 1, \cdots, \lfloor \log n_{k+1} \rfloor - 1$$

the c_n's being the Fourier coefficients of $f(x)$ as in (1). If all the numbers $\delta_\kappa^{(\mu)}$, are greater than $2\epsilon_k$, then

$$\sum_{\mu=0}^{[\log n_{k+1}]-1} \delta_k^{(\mu)} > 2\epsilon_k[\log n_{k+1}]$$

and at the same time

$$\sum_{\mu=0}^{[\log n_{k+1}]-1} \delta_k^{(\mu)} \leqq \log n_{k+1} \sum_{|n|=n_k+1}^{n_{k+1}} |c_n|^2, \quad \text{or} \quad 2\epsilon_k < \frac{\log n_{k+1}}{[\log n_{k+1}]} \epsilon_k.$$

From this contradiction, we may conclude that at least one of the numbers $\delta_k^{(\mu)}$, say $\delta_k^{(\mu_k)}$, does not exceed $2\epsilon_k$. We denote the corresponding Fourier coefficients by d_n: i.e.

$$d_n = \begin{cases} c_n, & n_k + \mu_k L_k + 1 \leqq |n| \leqq n_k + (\mu_k + 1)L_k, \\ 0, & \text{otherwise}, \quad k = 1, 2, \cdots. \end{cases}$$

It follows that

$$\sum_{n=-\infty}^{+\infty} |d_n|^2 \log |n| \leqq \sum_{k=1}^{\infty} \delta_k^{(\mu_k)} \leqq 2 \sum_{k=1}^{\infty} \epsilon_k < \infty.$$

Thus, the d_n's are the Fourier coefficients of a function $g(x)$ whose Fourier series converges almost everywhere [**1**, p. 253].

Now we define the sequence $\{m_\nu\}$ to take on the values m such that $n_k+\mu_k L_k+1 \leqq m \leqq n_k+(\mu_k+1)L_k$ for each k. Since the sequences

$$\{n_k + \mu_k L_k + 1\}, \ k = \text{odd}; \quad \text{and} \quad \{n_k + \mu_k L_k + 1\}, \quad k = \text{even}$$

are both lacunary, it follows from the Kolmogoroff theorem cited above that the sequence $s_{n_k+\mu_k L_k+1}(x;f)$ converges almost everywhere. Now our theorem follows from the fact that for each ν there is a k such that $n_k+\mu_k L_k+1 \leqq m_\nu \leqq n_k+(\mu_k+1)L_k$ and that we may write

$$s_{m_\nu}(x; f) = s_{n_k+\mu_k L_k+1}(x; f) + (s_{m_\nu}(x; g) - s_{n_k+\mu_k L_k+1}(x; g)),$$

the bracketed term on the right going to 0 almost everywhere.

It is not surprising to find that the case when $f(x)$ belongs to L^p, $1<p<2$, is more complicated, and our theorem here involves somewhat stricter hypotheses than for the L^2 case. The analogue for the number L_k will be $L_{k,p,\alpha}$ defined by

$$L_{k,p,\alpha} = \left[\frac{(n_{k+1} - n_k)^{2/p'}}{(\log n_{k+1})^{\alpha+1-2\alpha/p'}}\right], \quad \alpha > 0, \quad \frac{1}{p} + \frac{1}{p'} = 1.$$

Our theorem for the L^p case is the following.

THEOREM 2. *If $f(x)$ belongs to L^p, $1<p<2$, if $\{n_k\}$ is lacunary, and if $\sum_{k=1}^{\infty} 1/(\log n_k)^\alpha < \infty$ for some $\alpha>0$, then there is a sequence $\{m_\nu\}$ of positive integers containing $L_{k,p,\alpha}$ consecutive terms in each interval (n_k, n_{k+1}) such that the subsequence $s_{m_\nu}(x; f)$ of partial sums of the Fourier series of $f(x)$ converges almost everywhere to $f(x)$.*

We shall write $L_{k,p,\alpha}$ simply as L_k and let $M_k = [(n_{k+1} - n_k)/L_k]$. We introduce the following:

$$\epsilon_k = \sum_{|n|=n_k+1}^{n_{k+1}} |c_n|^{p'}; \quad \delta_k^{(\mu)} = \sum_{|n|=n_k+\mu L_k+1}^{n_k+(\mu+1)L_k} |c_n|^{p'}, \ \mu = 0, 1, \cdots, M_k - 1.$$

By Hölder's inequality,

$$\text{(2)} \qquad \log n_{k+1} \sum_{|n|=n_k+\mu L_k+1}^{n_k+(\mu+1)L_k} |c_n|^2 \leqq \log n_{k+1} (\delta_k^{(\mu)})^{2/p'} L_k^{(p'-2)/p'}.$$

The following is tentatively assumed:

$$\text{(3)} \qquad \epsilon_k > \frac{1}{(\log n_{k+1})^\alpha}.$$

If now

$$2\epsilon_k < (\log n_{k+1})(\delta_k^{(\mu)})^{2/p'} L_k^{(p'-2)/p'}$$

or

$$\text{(4)} \qquad \left(\frac{2\epsilon_k}{\log n_{k+1}}\right)^{p'/2} L_k^{(2-p')/2} < \delta_k^{(\mu)}, \qquad \mu = 0, 1, \cdots, M_k - 1,$$

then (assuming, as we may, that $M_k \geqq (n_{k+1} - n_k)/2L_k$)

$$2^{p'/2-1}(n_{k+1} - n_k)\left(\frac{\epsilon_k}{\log n_{k+1}}\right)^{p'/2} L_k^{-p'/2} < \sum_{\mu=0}^{M_k-1} \delta_k^{(\mu)} \leqq \epsilon_k.$$

Thus,

$$L_k > \frac{2^{1-2/p'}(n_{k+1} - n_k)^{2/p'} \epsilon_k^{1-2/p'}}{\log n_{k+1}} > \frac{2^{1-2/p'}(n_{k+1} - n_k)^{2/p'}}{(\log n_{k+1})^{1+\alpha-2\alpha/p'}}$$

by virtue of (3). This contradicts the definition of L_k so that (4) is false, and for some μ, say μ_k,

$$(\log n_{k+1})(\delta_k^{(\mu)})^{2/p'} L_k^{(p'-2)/p'} \leqq 2\epsilon_k.$$

From (2) it follows that

$$(5) \qquad (\log n_{k+1}) \sum_{|n|=n_k+\mu_k L_k+1}^{n_k+(\mu_k+1)L_k} |c_n|^2 \leqq 2\epsilon_k.$$

If (3) does not hold, i.e. if $\epsilon_k \leqq (\log n_{k+1})^{-\alpha}$, and if

$$(6) \quad (\log n_{k+1})(\delta_k^{(\mu)})^{2/p'} L_k^{(p'-2)/p'} > 2(\log n_{k+1})^{-\alpha}, \ \mu = 0, 1, \cdots, M_k - 1,$$

then

$$2^{p'/2} L_k^{1-2/p'} (\log n_{k+1})^{-(1+\alpha)p'/2} < \delta_k^{(\mu)}, \qquad \mu = 0, 1, \cdots, M_k - 1,$$

and

$$M_k 2^{p'/2} L_k^{1-p'/2} (\log n_{k+1})^{-(1+\alpha)p'/2} < \sum_{\mu=0}^{M_k-1} \delta_k^{(\mu)} \leqq \epsilon_k \leqq (\log n_{k+1})^{-\alpha}.$$

It follows from the preceding that

$$\frac{2^{1-2/p'}(n_{k+1} - n_k)^{2/p'}}{(\log n_{k+1})^{\alpha+1-2\alpha/p'}} < L_k.$$

This contradicts the definition of L_k so that (6) is false, and for some μ, say μ_k,

$$(\log n_{k+1})(\delta_k^{(\mu_k)})^{2/p'} L_k^{(p'-2)/p'} \leqq 2(\log n_{k+1})^{-\alpha}.$$

Combining this result with (5) we have

$$(7) \qquad (\log n_{k+1}) \sum_{|n|=n_k+\mu_k L_k+1}^{n_k+(\mu_k+1)L_k} |c_n|^2 \leqq 2(\epsilon_k + (\log n_{k+1})^{-\alpha}).$$

By hypothesis, $\sum_{k=1}^{\infty} (\log n_{k+1})^{-\alpha} < \infty$, and by the Hausdorff-Young theorem, $\sum_{k=1}^{\infty} \epsilon_k < \infty$. The coefficients d_n are defined as in the proof of Theorem 1. By (7), they are the Fourier coefficients of a function $g(x)$ whose Fourier series converges almost everywhere. The rest of the proof is the same as that for Theorem 1, except that we must cite the Littlewood-Paley generalization of the Kolmogoroff theorem [1, p. 255].

We remark that since $\{n_k\}$ is lacunary, the convergence of

$\sum_{k=1}^{\infty} 1/(\log n_k)^\alpha$ is assured for $\alpha > 1$, but assuming more about $\{n_k\}$, e.g. letting $\log n_k \geqq \lambda^k$, $\lambda > 1$, we may take α as close to 0 as we please.

3. Given a sequence $\{m_\nu\}$ of positive integers strictly increasing to $+\infty$, the function $\sigma(n)$ is defined to be the number of terms of the sequence $\{m_\nu\}$ less than or equal to n. We shall say that the sequence $\{m_\nu\}$ has upper density β if $\limsup (\sigma(n))/n = \beta$. With these definitions, we state our last theorem, the proof of which seems to work only for the L^2 case.

THEOREM 3. *If $f(x)$ belongs to L^2, then there is a sequence $\{m_\nu\}$ of upper density one such that the subsequence*

$$s_{m_\nu}(x; f), \qquad \nu = 1, 2, \cdots,$$

of partial sums of the Fourier series of $f(x)$ converges almost everywhere to $f(x)$.

We let $\{k_\mu\}$ be a sequence of positive integers such that k_μ divides $k_{\mu+1}$ and such that if $\lambda_\mu = k_{\mu+1}/k_\mu$, then λ_μ increases strictly to $+\infty$. Now we define the following:

$$n_k = (\lambda_\mu)^k,\ k_\mu < k \leqq k_{\mu+1};\ \epsilon_k = \sum_{|n|=n_k+1}^{n_{k+1}} |c_n|^2,\ k = k_\mu + 1, \cdots, k_{\mu+1} - 1$$

$$D_\mu = \sum_{k=k_\mu+1}^{k_{\mu+1}-1} \epsilon_k.$$

If, for a given μ,

$$2D_\mu < \epsilon_k \log n_{k+1}, \qquad k = k_\mu + 1, \cdots, k_{\mu+1} - 1,$$

then

$$2 \sum_{k=k_\mu+1}^{k_{\mu+1}-1} \frac{D_\mu}{\log n_{k+1}} < \sum_{k=k_\mu+1}^{k_{\mu+1}-1} \epsilon_k = D_\mu$$

or

$$\frac{2}{\log \lambda_\mu} \sum_{k=k_\mu+1}^{k_{\mu+1}-1} \frac{1}{k+1} < 1. \tag{8}$$

But the sum on the left side of (8) is not less than $2^{-1} \log (k_{\mu+1}/k_\mu)$ for μ sufficiently large. Since $k_{\mu+1}/k_\mu = \lambda_\mu$, a contradiction is reached from which we may conclude that for each μ, there is a k, say $k(\mu)$, $k_\mu + 1 \leqq k(\mu) \leqq k_{\mu+1} - 1$, for which

$$\sum_{|n|=n_{k(\mu)}+1}^{n_{k(\mu)+1}} (\log |n|) |c_n|^2 \leqq \epsilon_{k(\mu)} \log n_{k(\mu)+1} \leqq 2D_\mu.$$

We choose the corresponding Fourier coefficients to define as before a new function $g(x)$, whose Fourier series converges almost everywhere since $\sum_{\mu=1}^{\infty} D_\mu < \infty$. Now we define $\{m_\nu\}$ to take on the values m, $n_{k(\mu)} < m \leqq n_{k(\mu)+1}$ for each μ. Since the sequence $\{n_{k(\mu)}\}$ is lacunary, the almost everywhere convergence of $s_{m_\nu}(x; f)$ to $f(x)$ follows as before. For the sequence $\{m_\nu\}$,

$$\frac{\sigma(n_{k(\mu)+1})}{n_{k(\mu)+1}} \geqq \frac{n_{k(\mu)+1} - n_{k(\mu)}}{n_{k(\mu)+1}} = 1 - \frac{1}{\lambda_\mu}$$

for each μ. Since the limit of the right side is 1, the theorem is proved.

Reference

1. A. Zygmund, *Trigonometrical series*, Warsaw, 1935.

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ON THE LOGARITHMIC MEAN OF THE DERIVED CONJUGATE SERIES OF A FOURIER SERIES

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1. Let $f(t)$ be integrable L in $(-\pi, \pi)$ and periodic with period 2π and let

$$(1.1)\quad f(t) \sim \frac{1}{2} a_0 + \sum_1^\infty (a_n \cos nt + b_n \sin nt) = \frac{1}{2} a_0 + \sum_1^\infty A_n(t).$$

The differentiated conjugate series of (1.1) at $t = x$ is

$$(1.2)\qquad -\sum_1^\infty x(a_n \cos nx + b_n \sin nx) = -\sum_1^\infty nA_n(x).$$

We write

$$\phi(t) = f(x+t) + f(x-t) - 2f(x), \qquad h(t) = \frac{\phi(t)}{4 \sin \frac{1}{2}t} - d,$$

where d is a function of x.

Let S_n, t_n, and σ_n be the nth partial sum, the first Cesàro mean, and the first logarithmic mean of the series (1.2) respectively. The

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