A STRONG LIMIT THEOREM FOR GAUSSIAN PROCESSES

GLEN BAXTER

1. Introduction and main result. Let $\{W(t),\ 0 \leqq t \leqq 1\}$ be the Wiener process, that is, a Gaussian stochastic process of real-valued random variables with $E\{W(t)\} = 0$ and $E\{W(s)W(t)\} = \min\{s, t\}$. It was discovered independently by Levy [4] and by Cameron and Martin [1] that with probability one

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \left[W\left(\frac{k}{2^n}\right) - W\left(\frac{k-1}{2^n}\right)\right]^2 = 1. \tag{1}$$

We demonstrate here a similar result for a large class of Gaussian processes including the Wiener process as a particular case. Notation: in the following $\{X(t),\ 0 \leqq t \leqq 1\}$ will denote a Gaussian stochastic process of real-valued random variables with mean function $E\{X(t)\} = m(t)$ and covariance function $E\{X(s)X(t)\} - m(s)m(t) = r(s, t)$.

Now assume that $m(t)$ has a bounded first derivative for $0 \leqq t \leqq 1$. Furthermore, assume that $r(s, t)$ is continuous in $0 \leqq s,\ t \leqq 1$ and has uniformly bounded second derivatives for $s \neq t$. Let

$$D^+(t) = \lim_{s\to t^+} \frac{r(t, t) - r(s, t)}{t - s},$$

$$D^-(t) = \lim_{s\to t-} \frac{r(t, t) - r(s, t)}{t - s},$$

$$f(t) = D^-(t) - D^+(t).$$

The uniform boundedness of the second order derivatives of $r(s, t)$ implies the existence, boundedness, and continuity of the functions $D^+(t)$, $D^-(t)$, and $f(t)$ over $0 < t < 1$. In particular $f(t)$ is Riemann integrable over the interval $0 < t < 1$. The main result can now be stated.

THEOREM 1. *If $\{X(t),\ 0 \leqq t \leqq 1\}$ is a Gaussian process satisfying the assumptions of the preceding paragraph, then with probability one*

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 f(t)dt. \tag{2}$$

Note that the existence of the first derivative of $r(s, t)$ at $s = t$ is not

Received by the editors May 27, 1955.

assumed, in fact, it is not wanted. The existence of this derivative would make $f(t)\equiv 0$ and the right-hand side of (2) would reduce to zero, giving an uninteresting result. An important corollary to Theorem 1 holds in case $r(s, t)$ factors into the product of a function of s and a function of t. As we shall see later there are many examples of Gaussian processes for which the covariance function has this property.

COROLLARY. *If* $\{X(t), 0\leqq t\leqq 1\}$ *satisfies the assumptions of Theorem* 1, *and if*

$$r(s, t) = \begin{cases} u(s)v(t), & s \leqq t, \\ u(t)v(s), & s \geqq t, \end{cases} \tag{3}$$

then with probability one

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 \{u'(t)v(t) - u(t)v'(t)\}\,dt. \tag{4}$$

The proof of Theorem 1 is contained in the next section. In §3 we give some examples of the theorem and show that (1) is implied by (2) (and by (4)).

2. **Proof of Theorem 1.** Let us first assume that $m(t)\equiv 0$. For any positive integer n and integers j, k with $1\leqq j$, $k\leqq 2^n$ set

$$\Delta X_k = X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right),$$

$$a_{jk} = E\{\Delta X_j \Delta X_k\},$$

$$B_n = \sum_{k=1}^{2^n} \Delta X_k^2,$$

$$b_n = 2\sum_{j,k=1}^{2^n} a_{jk}^2.$$

Observe that

$$E\{B_n\} = \sum_{k=1}^{2^n} a_{kk} \tag{5}$$

while

$$E\{B_n^2\} = \sum_{k=1}^{2^n} 3a_{kk}^2 + 2\sum_{j>k=1}^{2^n} (a_{kk}a_{jj} + 2a_{jk}^2).$$

Thus the variance of B_n is exactly b_n. Applying Tchebychev's inequality

$$P\{\,|\,B_n - E\{B_n\}\,|\, > n/(2^n)^{1/2}\} < 2^n b_n/n^2.$$

If we can show that $2^n b_n$ remains bounded as n becomes infinite, an application of the Borel-Cantelli lemma will show that with probability one $B_n - E\{B_n\}$ approaches zero as n becomes infinite.

To estimate $2^n b_n$, let M be a bound for the three quantities $|\partial^2 r(s, t)/\partial s^2|$, $|\partial^2 r(s, t)/\partial s\partial t|$, and $|\partial^2 r(s, t)/\partial t^2|$ in the range $0 \leqq s \neq t \leqq 1$. Using for $r(s, t)$ a Taylor series expansion with remainder it can easily be shown that $j \neq k$ implies

$$\begin{aligned} |\,a_{jk}\,| &= |\,r(k/2^n, j/2^n) + r((k-1)/2^n, (j-1)/2^n) \\ &\quad - r(k/2^n, (j-1)/2^n) - r((k-1)/2^n, j/2^n)\,| \leqq 3M(1/2)^{2n}. \end{aligned} \tag{6}$$

Also for $k = 1, 2, \cdots, 2^n - 1$

$$\begin{aligned} a_{kk} &= r(k/2^n, k/2^n) - 2r((k-1)/2^n, k/2^n) \\ &\qquad + r((k - 1/2^n), (k-1)/2^n) \\ &= (1/2)^n[D^-(k/2^n) - D^+(k/2^n)] + O((1/2)^{2n}) \end{aligned} \tag{7}$$

where $2^{2n}O((1/2)^{2n})$ remains bounded independent of k as n becomes infinite. The estimates in (6) and (7) give the boundedness of $2^n b_n$. In conjunction with (5), the estimate in (7) also implies that

$$E\{B_n\} = \sum_{k=1}^{2^n-1} \left(\frac{1}{2}\right)^n f(k/2^n) + O\left(\left(\frac{1}{2}\right)^n\right)$$

so that $E\{B_n\}$ approaches a limit as n becomes infinite. This limit is exactly the Riemann integral of $f(t)$ from 0 to 1. Writing

$$B_n = E\{B_n\} + [B_n - E\{B_n\}]$$

we see that with probability one B_n itself converges to the integral of $f(t)$ over $(0, 1)$. This finishes the case for which $m(t) \equiv 0$.

If $m(t) \not\equiv 0$, form a new Gaussian process $\{\overline{X}(t), 0 \leqq t \leqq 1\}$ by taking $\overline{X}(t) = X(t) - m(t)$. From the previous arguments we know that with probability one

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \Delta\overline{X}_k^2 = \int_0^1 f(t)dt.$$

Now by Schwarz's inequality with $\Delta m_k = m(k/2^n) - m((k-1)/2^n)$

$$\left[\sum_{k=1}^{2^n} \Delta m_k \Delta\overline{X}_k\right]^2 \leqq \sum_{k=1}^{2^n} \Delta m_k^2 \cdot \sum_{k=1}^{2^n} \Delta\overline{X}_k^2. \tag{8}$$

The second term on the right-hand side of (8) remains bounded with probability one. Since $m'(t)$ exists and is uniformly bounded over (0, 1), the sum of the Δm_k^2 goes to zero as n becomes infinite. Thus, with probability one the left-hand side of (8) goes to zero. Finally,

$$\sum_{k=1}^{2^n} \Delta X_k^2 = \sum_{k=1}^{2^n} \Delta \overline{X}_k^2 + 2 \sum_{k=1}^{2^n} \Delta m_k \Delta \overline{X}_k + \sum_{k=1}^{2^n} \Delta m_k^2$$

so that

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \Delta X_k^2 = \lim_{n\to\infty} \sum_{k=1}^{2^n} \Delta \overline{X}_k^2 = \int_0^1 f(t)dt.$$

This ends the proof of Theorem 1.

3. **Examples.** (a) Let us first show that (1) is included in (2) and (4). For the Wiener process $r(s, t)$ factors into a product of a function of s and a function of t like (3); in fact,

$$r(s, t) = \begin{cases} s, & s \leqq t, \\ t, & s \geqq t. \end{cases}$$

In the notation of the corollary to Theorem 1, $u(s) = s$ and $v(t) = 1$ so (4) states that with probability one

$$\lim_{n\to\infty} \sum_{k=1}^{2^n} \left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 1dt = 1.$$

(b) For a second example[1] we consider a class of Gaussian processes whose covariance functions are actually Green's functions of certain simple boundary value problems.

It is necessary and sufficient in order that $r(s, t)$ defined for $0 \leqq s, t \leqq 1$ be the covariance function of a Gaussian process $\{X(t), 0 \leqq t \leqq 1\}$ that

(i) $r(s, t) = r(t, s)$;

(ii) if $t_1, t_2, \cdots, t_n \in [0, 1]$, then $(r(t_i, t_j))$ is a non-negative definite matrix.

In case $r(s, t)$ has the form (3) where both $u(t)$ and $v(t)$ are non-negative for $0 \leqq t \leqq 1$, these two conditions can be reduced to just

(iii) if $0 \leqq t_1 < t_2 \leqq 1$, then $u(t_2)v(t_1) - u(t_1)v(t_2)$ is non-negative.

To verify this last statement we need only mention that if $r(s, t)$ has the form (3), (i) is automatically satisfied while the determinant of $(r(t_i, t_j))$ can be explicitly evaluated:

[1] Some of the results here are taken from the author's doctoral dissertation which was written under the guidance of Professor Monroe Donsker at the University of Minnesota.

$$\left| (r(t_i, t_j)) \right| = u(t_1)v(t_n) \prod_{i=2}^{n} \{u(t_i)v(t_{i-1}) - u(t_{i-1})v(t_i)\}.$$

If (iii) holds the determinant of $(r(t_i, t_j))$ is non-negative for any choice of $t_1, \cdots, t_n \in [0, 1]$. This shows (see [5, p. 103]) that $(r(t_i, t_j))$ is non-negative definite. On the other hand, if (ii) holds for $r(s, t)$ of the form (3), then the implications of (ii) in the case $0 \leqq t_1 < t_2 \leqq 1$ give (iii).

We now show how covariance functions which factor like (3) may come from certain differential systems.

THEOREM 2. *Let $p(s) > 0$, $q(s) \geqq 0$, and $p'(s)$ be continuous functions on $0 \leqq s \leqq 1$, and let h and H be non-negative extended real-valued numbers. If*

$$(9) \qquad \begin{cases} \dfrac{d}{ds}\left\{p(s)\dfrac{dy}{ds}\right\} - q(s)y = 0, \\ y(0) - hy'(0) = 0, \\ y(1) + Hy'(1) = 0 \end{cases}$$

is incompatible, then the Green's function $r(s, t)$ (see [2, p. 304]) is a covariance function of a Gaussian process.

PROOF. It is shown in the reference above that the Green's function of system (9) has the form (3) where $u(s)$ and $v(s)$ are linearly independent solutions of the differential equation and satisfy, respectively, the boundary conditions at $s=0$ and $s=1$. We must show that $r(s, t)$ is non-negative and that $u(t)v(s) - u(s)v(t) \geqq 0$ for $s \leqq t$.

To show that $r(s, t)$ is non-negative write

$$p(s)u'(s) - p(0)u'(0) = \int_0^s u(x)q(x)dx.$$

By the non-negativity of $p(s)$ and $q(s)$ we deduce that $u(s)$ cannot change sign in the interval $0 \leqq s \leqq 1$ and is increasing or decreasing according as it is positive or negative. Similarly we deduce that $v(s)$ does not change sign in the interval $0 \leqq s \leqq 1$ and is increasing or decreasing according as it is negative or positive. Suppose for a while that $u(t)v(s) - u(s)v(t)$ is positive whenever $t > s$. If $u(s)v(t)$ is negative, then either $u(s)$ is positive and $v(t)$ is negative or vice-versa. Since either case is handled in the same way we consider only $u(s)$ positive and $v(t)$ negative. In that case $t > s$ implies $u(t)v(t) \leqq u(s)v(t) < u(t)v(s)$ which contradicts the fact that $v(t)$ is increasing in t. Thus $u(s)v(t)$ is non-negative for $0 \leqq s, t \leqq 1$.

To prove that $u(t)v(s) - u(s)v(t)$ is positive for $t > s$ we write

$$p(t)\{u'(t)v(s) - u(s)v'(t)\} - 1 \tag{10}$$
$$= \int_s^t q(x)\{u(x)v(s) - v(x)u(s)\}dx.$$

For t in the neighborhood of s the right-hand side of (10) is very small. Thus the derivative of $u(t)v(s)-u(s)v(t)$ with respect to t (fixed but arbitrary s) is positive for $t>s$ so that the positivity of $u(t)v(s)-u(s)v(t)$ $(t>s)$ is verified. This completes the proof of the theorem.

Now let $\{X(t),\ 0\leqq t\leqq 1\}$ be a Gaussian stochastic process whose covariance function is the Green's function of (9). By the definition of the Green's function $u'(t)v(t)-u(t)v'(t)=1/p(t)$ so that the corollary to Theorem 1 states that

$$\lim_{n\to\infty} \sum_{k=1}^{2^n}\left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 \frac{1}{p(t)}\,dt \tag{11}$$

with probability one. For an explicit example take the system (9) where $p(s)=1$, and $q(s)=h=H=0$. The system is incompatible and the Green's function is

$$r(s, t) = \begin{cases} s(1 - t), & s \leqq t, \\ t(1 - s), & s \geqq t. \end{cases}$$

The Gaussian process with this covariance function and mean function zero has been the subject of considerable study by Doob [3] and others. We have immediately for this process

$$\lim_{n\to\infty} \sum_{k=1}^{2^n}\left[X\left(\frac{k}{2^n}\right) - X\left(\frac{k-1}{2^n}\right)\right]^2 = \int_0^1 \frac{1}{1}\,dt = 1.$$

It is easy to see that the Wiener process is another explicit example of Theorem 2 and (11).

References

1. R. H. Cameron and W. T. Martin, *The behavior of measure and measurability under a change of scale in Wiener space*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 130–137.

2. R. Courant and D. Hilbert, *Methoden der Mathematischen Physik*, vol. 1, Berlin, Springer, 1931.

3. J. L. Doob, *Heuristic approach to the Kolmogorov-Smirnov theorems*, Ann. Math. Statist. vol. 20 (1949) pp. 393–403.

4. P. Levy, *Le mouvement brownien plan*, Amer. J. Math. vol. 62 (1940) pp. 487–550.

5. S. Perlis, *Theory of matrices*, Cambridge, Addison-Wesley, 1952.

Massachusetts Institute of Technology