UNIVERSAL INFINITE PARTIALLY ORDERED SETS

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Introduction. The purpose of this paper is to construct for each infinite cardinal $\aleph$ an $\aleph$-universal partially ordered set in which every partially ordered set of cardinality $\aleph$ can be embedded. An $\aleph$-universal partially ordered set of cardinality $2^\aleph$ is constructed for every infinite cardinal $\aleph$. Furthermore, an $\aleph$-universal partially ordered set of cardinality $\aleph$ is constructed for every cardinal $\aleph$ which is the sum of denumerably many smaller cardinals. The existence for an arbitrary cardinal $\aleph$ of an $\aleph$-universal partially ordered set of cardinality $\aleph$ is left an open question and appears to be a difficult problem.

The ordering relation "$<$" in a partially ordered set is understood to be transitive, nonreflexive, and nonsymmetric. The Axiom of Choice will be assumed throughout this paper. The General Continuum Hypothesis will be explicitly mentioned when it is used.

1. Totally ordered extensions of partially ordered sets.

DEFINITION:[1] Let $\alpha \in On$. If the functions $f: \bar{\omega}_\alpha \to \bar{2}$ are ordered according to first differences, the set $\overline{2^{\bar{\omega}\alpha}}$ becomes a totally ordered set which we denote by $T(\overline{2^{\bar{\omega}\alpha}})$. We shall call a function $f: \bar{\omega}_\alpha \to \bar{2}$ an *α-rational* if there exists an ordinal $\eta(f) < \omega_\alpha$, necessarily unique, such that $f'\eta(f) = 1$ while $f'\mu = 0$ for $\eta(f) < \mu < \omega_\alpha$. In the remainder of this paper η will be used exclusively to denote this "terminating" ordinal of the α-rationals. We denote by T_α the totally ordered suborder of $T(\overline{2^{\bar{\omega}\alpha}})$ consisting of the α-rationals.

THEOREM 1. *Let $\alpha \in On$. T_α is a totally ordered set of cardinality $\leq 2^{\aleph_\alpha}$. Let $\rho < \omega_\alpha$ and let A be any subset of T_α such that for every $f \in A$, $\eta(f) < \rho$. Then there exists a (unique) minimal element $h(A, \rho) \in T_\alpha$ having the properties that $\eta[h(A, \rho)] = \rho$ and $h(A, \rho) > f$ in T_α for every $f \in A$. If $g \in T_\alpha$ with $\eta(g) < \rho$ and $g > f$ in T_α for every $f \in A$, then $g > h(A, \rho)$ in T_α. If $\alpha = 0$ or if the General Continuum Hypothesis holds, T_α has cardinality $\aleph_\alpha$.*

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[1] *On* denotes the class of ordinals. If μ is an ordinal, $\bar{\mu}$ denotes the class of ordinals less than μ. To show that a function F has a set R for its domain and has its range in a set S, we write $F: R \to S$. For particular sets R and S, the set of all functions $F: R \to S$ is denoted by S^R. The image of an element a by a function F is denoted by $F'a$.

Proof. (1) $T_\alpha = \bigcup \{B(\eta): \eta < \omega_\alpha\}$, where $B(\eta) = \{f: f \in T_\alpha$ and $\eta(f) = \eta\}$ and the $B(\eta)$ are disjoint, so that $|T_\alpha| = \sum \{|B(\eta)| : \eta < \omega_\alpha\}$. Now $|B(\eta)| = 2^{|\eta|} \leq 2^{\aleph_\alpha}$, from which we infer that $|T_\alpha| \leq 2^{\aleph_\alpha}$. If $\alpha = 0$ or if the General Continuum Hypothesis holds, then $|B(\eta)| \leq \aleph_\alpha$, and we obtain $|T_\alpha| = \aleph_\alpha$.

(2) Let ρ and A satisfy the hypothesis of the theorem. We take for $h(A, \rho)$ the unique function $k: \bar\omega_\alpha \to \bar 2$ defined inductively by:

(a) for $\mu < \rho$, $k'\mu = 1$ if and only if there exists an $f \in A$ such that $f|\bar\mu = k|\bar\mu$ and $f'\mu = 1$;[2]

(b) $k'\rho = 1$ and $k'\mu = 0$ for $\rho < \mu < \omega_\alpha$.

Clearly, $k \in T_\alpha$ and $\eta(k) = \rho$.

(3) Let $f \in A$. Since $\eta(f) < \rho = \eta(k)$, there exists a first ordinal $\sigma \leq \rho$ for which $f'\sigma \neq k'\sigma$ while $f|\bar\sigma = k|\bar\sigma$. If it were true that $f'\sigma = 1 > 0 = k'\sigma$, then necessarily $\sigma < \rho$ and we would have a contradiction of (2). Thus $f'\sigma = 0 < 1 = k'\sigma$ and $f < k$ in T_α.

(4) Let $g \in T_\alpha$ with $\eta(g) = \tau \leq \rho = \eta(k)$ such that $g < k$ in T_α. Then there exits a first ordinal $\sigma \leq \rho$ such that $g'\sigma = 0 < 1 = k'\sigma$ while $g|\bar\sigma = k|\bar\sigma$. Note that not both $\sigma = \rho$ and $\tau = \rho$.

(a) Suppose that $\sigma < \rho$. Then by (2), $k'\sigma = 1$ implies that there exists an $f \in A$ such that $f|\bar\sigma = k|\bar\sigma = g|\bar\sigma$ while $f'\sigma = k'\sigma = 1 > 0 = g'\sigma$. For this $f \in A$ we have $f > g$ in T_α.

(b) Suppose that $\tau < \sigma = \rho$. Then by (2), $k'\tau = g'\tau = 1$ implies that there exists an $f \in A$ such that $f|\bar\tau = k|\bar\tau = g|\bar\tau$ while $f'\tau = k'\tau = g'\tau = 1$. Since $\eta(g) = \tau$, we have $g \leq f$ in T_α for this particular f.

The minimality of $h(A, \rho) = k$ follows from (a) with $\tau = \rho$. For $g \in T_\alpha$ with $\eta(g) < \rho$ and $g > f$ in T_α for every $f \in A$, the fact that $g > k$ in T_α follows from (b) and (a) with $\tau < \rho$, since $\eta(g) < \rho$ implies that $g \neq k$. This completes the proof of Theorem 1.

If P is a partially ordered set and T a totally ordered set, a bi-unique function $F: P \to T$ is said to *generate a totally ordered extension* of P if for any two elements a and b of P, $a < b$ in P implies that $F'a < F'b$ in T. The totally ordered extension generated is the set of elements of P together with the order relations transferred from T back to P by the inverse of F. If P is itself a totally ordered set, then P is similar to the totally ordered set $\{F'a: a \in P\}$ and F is called an *embedding* of P in T.

Theorem 2. *Let* $\alpha \in On$ *and let* P *be any partially ordered set of cardinality* $\aleph_\alpha$ *with an imposed well-ordering,* $P = \{p(\mu): \mu < \omega_\alpha\}$. *For* $\nu \leq \omega_\alpha$, *define* $P_\nu = \{p(\mu): \mu < \nu\}$. *Let* $0 \leq \rho < \sigma \leq \omega_\alpha$ *and suppose that* G:

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[2] If F is a function and R is a subset of its domain, $F|R$ denotes the function F restricted to the domain R.

$P_\rho \to T_\alpha$ *is any function which generates a totally ordered extension of* P_ρ *in such a manner that for each* $p(\mu) \in P_\rho$, $\eta[G'p(\mu)] = \mu$. *By induction there exists a unique function* $F \colon P_\sigma \to T_\alpha$ *such that* $F|P_\rho = G$ *while* $F'p(\nu) = h(A_\nu, \nu)$ *for* $\rho \leq \nu < \sigma$, *where* $A_\nu = \{F'p(\mu) \colon \mu < \nu$ *and* $p(\mu) < p(\nu)$ *in* $P\}$. *We conclude that this function* F *generates a totally ordered extension of* P_σ *in such a manner that for each* $p(\mu) \in P_\sigma$, $\eta[F'p(\mu)] = \mu$. *We denote the function* $F|(P_\sigma - P_\rho)$ *by* $F(G, \sigma)$.

PROOF. (1) Let $\mu, \nu < \rho$ with $p(\mu) < p(\nu)$ in P. Then $F'p(\mu) = G'p(\mu) < G'p(\nu) = F'p(\nu)$ in T_α by the assumed property of G.

(2) Let $\mu < \nu < \sigma$ with $\nu \geq \rho$ and $p(\mu) < p(\nu)$ in P. Then $F'p(\mu) \in A_\nu$ so that $F'p(\nu) = h(A_\nu, \nu) > F'p(\mu)$ in T_α by Theorem 1.

(3) Let $\mu < \nu < \sigma$ with $\nu \geq \rho$ and $p(\mu) > p(\nu)$ in P. Suppose it were true that $F'p(\mu) < F'p(\nu)$ in T_α. Let τ be the first such ordinal ν. Then $p(\mu) > p(\tau)$ in P implies that $p(\mu) > p(\pi)$ in P for every $\pi < \tau$ for which $p(\tau) > p(\pi)$ in P. If $\pi < \mu$ then $F'p(\pi) < F'p(\mu)$ in T_α by (1) and (2). If $\mu < \pi < \rho$ then by (1), $F'p(\pi) < F'p(\mu)$ in T_α. If $\mu < \pi < \tau$ and $\pi \geq \rho$ then by the minimality of τ, $F'p(\pi) < F'p(\mu)$ in T_α. Thus for every $f \in A_\tau$ we have $F'p(\mu) > f$ in T_α so that by Theorem 1, $F'p(\mu) > h(A_\tau, \tau) = F'p(\tau)$ in T_α, contrary to our supposition. So, for $\mu < \nu < \sigma$ with $\nu \geq \rho$ and $p(\mu) > p(\nu)$ in P, we have $F'p(\mu) > F'p(\nu)$ in T_α, which completes the proof of Theorem 2.

For each $\alpha \in On$ we say that a totally ordered set T is $\aleph_\alpha$-*universal* if every totally ordered set of cardinality $\aleph_\alpha$ has an embedding in T.

If in Theorem 2 we take P to be any totally ordered set of cardinality $\aleph_\alpha$ and set $0 = \rho$ and $\sigma = \omega_\alpha$, we obtain immediately the theorem of N. Cuesta Dutari [2, p. 243, Theorem 15]: *For each* $\alpha \in On$, T_α *is an* $\aleph_\alpha$-*universal totally ordered set of cardinality* $\leq 2^{\aleph_\alpha}$; *if* $\alpha = 0$ *or if the General Continuum Hypothesis holds then* T_α *has cardinality* $\aleph_\alpha$. This theorem has also been obtained for nonlimit cardinals $\aleph_\alpha$ by W. Sierpiński [6, p. 62, Theorem 3], using a theorem due to F. Hausdorff [3, pp. 181–182]; the method of proof could be extended to regular cardinals but not to singular cardinals.

Suppose that P is a partially ordered set of cardinality $\aleph_\alpha$ having distinct noncomparable elements a and b. If we impose on P a well-ordering, $P = \{p(\mu) \colon \mu < \omega_\alpha\}$, in which $p(0) = b$ and $p(1) = a$, and if we set $0 = \rho$ and $\sigma = \omega_\alpha$, then Theorem 2 yields immediately the theorem of E. Szpilrajn [7]: *If* P *is a partially ordered set having distinct noncomparable elements* a *and* b, *then there exists a totally ordered extension of* P *in which* $a < b$.

2. **Universal infinite partially ordered sets.** If P and Q are partially ordered sets, a bi-unique function $F \colon P \to Q$ is called an *embedding* of

P in Q if for every two elements a and b of P, $a < b$ in P if and only if $F'a < F'b$ in Q.

Let P be a partially ordered set. A set K of totally ordered extensions of P is said to *realize* P if for any two elements a and b of P, $a < b$ in P if and only if $a < b$ in every member of K. By the theorem of E. Szpilrajn [7], the set of all totally ordered extensions of P realizes P. Thus, among such realizing sets K there will be one of minimal cardinality (≥ 1). B. Dushnik and E. W. Miller [1] call this minimal cardinal the *dimension* of P and prove the following statements:

(*) Every infinite partially ordered set has a dimension less than or equal to its cardinality.

(**) If a partially ordered set P has an embedding in a partially ordered set Q, then dimension $P \leq$ dimension Q.

(***) For $\alpha \in On$ and any cardinal $\mathfrak{m}$ satisfying $0 < \mathfrak{m} \leq \aleph_\alpha$, there exists a partially ordered set of cardinality $\aleph_\alpha$ and dimension $\mathfrak{m}$.

For each $\alpha \in On$, we say that a partially ordered set P is $\aleph_\alpha$-*universal* if every partially ordered set of cardinality $\aleph_\alpha$ has an embedding in P; we say that P is $\aleph_\alpha$-*sub-universal* if every partially ordered set of cardinality $\aleph_\alpha$ and dimension $< \aleph_\alpha$ has an embedding in P.

DEFINITION. Let $\alpha \in On$. The set $T_\alpha^{\bar{\omega}\alpha}$ clearly becomes a partially ordered set, which we denote by $P(T_\alpha^{\bar{\omega}\alpha})$, if we order the functions $F: \bar{\omega}_\alpha \to T_\alpha$ as follows: for $F_1, F_2 \in T_\alpha^{\bar{\omega}\alpha}$, $F_1 < F_2$ in $P(T_\alpha^{\bar{\omega}\alpha})$ if and only if for every $\mu < \omega_\alpha$, $F_1'\mu < F_2'\mu$ in T_α. A function $F: \bar{\omega}_\alpha \to T_\alpha$ is said to be of *constant degree* if there exists an ordinal $\delta < \omega_\alpha$ such that for every $\mu < \omega_\alpha$, $\eta[F'\mu] = \delta$; this ordinal δ is denoted by $\Delta(F)$. A function $F: \bar{\omega}_\alpha \to T_\alpha$ is said to be *periodic* if there exists an ordinal π with $0 < \pi < \omega_\alpha$ such that for every $\mu < \omega_\alpha$, $F'\mu = F'\Phi(\pi, \mu)$;[3] the minimal such π is denoted by $\prod(F)$. $P(\aleph_\alpha)$ will denote the partially ordered suborder of $P(T_\alpha^{\bar{\omega}\alpha})$ consisting of the functions of constant degree. P_α will denote the partially ordered suborder of $P(\aleph_\alpha)$ consisting of the periodic functions of constant degree.

THEOREM 3. *Let $\alpha \in On$. Then $P(\aleph_\alpha)$ is an $\aleph_\alpha$-universal partially ordered set of cardinality $2^{\aleph_\alpha}$ and dimension $\aleph_\alpha$. P_α is an $\aleph_\alpha$-sub-universal partially ordered set of cardinality $\leq 2^{\aleph_\alpha}$ for which $\lim \{\mathfrak{m}: \mathfrak{m} < \aleph_\alpha\} \leq$ dimension $P_\alpha \leq \aleph_\alpha$; if $\alpha = 0$ or if the General Continuum Hypothesis holds, then P_α has cardinality $\aleph_\alpha$.*

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[3] If μ and ν are ordinals with $\mu > 0$, then there exist unique ordinals $\Theta(\mu, \nu)$ and $\Phi(\mu, \nu)$ with $\Theta(\mu, \nu) \leq \nu$ and $\Phi(\mu, \nu) < \mu$ such that ν has the representation: $\nu = \mu \times \Theta(\mu, \nu) + \Phi(\mu, \nu)$.

PROOF. (1) $P(\aleph_\alpha) = \bigcup \{B(\delta): \delta < \omega_\alpha\}$, where $B(\delta) = \{F: F \in P(\aleph_\alpha)$ and $\Delta(F) = \delta\}$ and the $B(\delta)$ are disjoint, so that $|P(\aleph_\alpha)| = \sum \{|B(\delta)|: \delta < \omega_\alpha\}$. Now $|B(\delta)| = 2^{|\delta| \times \aleph_\alpha} = 2^{\aleph_\alpha}$ for $\delta > 0$, so that $|P(\aleph_\alpha)| = 2^{\aleph_\alpha}$.

(2) $P_\alpha = \bigcup \{B(\delta, \pi): \delta < \omega_\alpha, \pi < \omega_\alpha\}$, where $B(\delta, \pi) = \{F: F \in P_\alpha, \Delta(F) = \delta, \text{ and } \prod(F) = \pi\}$ and the $B(\delta, \pi)$ are disjoint, so that $|P_\alpha| = \sum \{|B(\delta, \pi)|: \delta < \omega_\alpha, \pi < \omega_\alpha\}$. Now $|B(\delta, \pi)| = 2^{|\delta| \times |\pi|} \leq 2^{\aleph_\alpha}$, so that $|P_\alpha| \leq 2^{\aleph_\alpha}$. If $\alpha = 0$ or if the General Continuum Hypothesis holds, then $|B(\delta, \pi)| \leq \aleph_\alpha$ and we obtain $|P_\alpha| = \aleph_\alpha$.

(3) Let P be any partially ordered set of cardinality $\aleph_\alpha$ with an imposed well-ordering, $P = \{p(\mu): \mu < \omega_\alpha\}$. Let K be a realizing set of totally ordered extensions of P with $|K| = \text{dimension } P$ so that we may write $K = \{T(\nu): \nu < \zeta\}$, where ζ is the initial ordinal of dimension P and $\zeta \leq \omega_\alpha$ by (*). For $\nu < \zeta$, define $G(\nu): T(\nu) \to T_\alpha$ to be the function F of Theorem 2 for the totally ordered set $T(\nu) = \{p(\mu): \mu < \omega_\alpha\}$, taking $0 = \rho$ and $\sigma = \omega_\alpha$. If $\zeta < \omega_\alpha$, define $G(\nu) = G[\Phi(\zeta, \nu)]$ for $\zeta \leq \nu < \omega_\alpha$. For each $\mu < \omega_\alpha$ we define $F(\mu): \bar{\omega}_\alpha \to T_\alpha$ by setting $F(\mu)'\nu = G(\nu)'p(\mu)$ for each $\nu < \omega_\alpha$. Finally we define $F: P \to P(T_\alpha^{\bar{\omega}_\alpha})$ by setting $F'p(\mu) = F(\mu)$ for each $\mu < \omega_\alpha$. By the assumed property of K and the above constructions, F is an embedding of P in $P(\aleph_\alpha)$; furthermore, if dimension $P < \aleph_\alpha$, then $\zeta < \omega_\alpha$ and F is an embedding of P in P_α.

(4) By (**), (***), and step (3) above, it remains only to exhibit a set of $\aleph_\alpha$ totally ordered extensions of $P(\aleph_\alpha)$ which realizes $P(\aleph_\alpha)$. We define totally ordered extensions $L(\mu)$, $\mu < \omega_\alpha$, of $P(\aleph_\alpha)$ as follows. For distinct elements F_1 and F_2 of $P(\aleph_\alpha)$ we set $F_1 < F_2$ in $L(\mu)$ if and only if:

(a) $F_1'\mu < F_2'\mu$ in T_α or,

(b) $F_1'\mu = F_2'\mu$ and $F_1'\nu > F_2'\nu$ in T_α for the first ordinal $\nu < \omega_\alpha$ for which $F_1'\nu \neq F_2'\nu$.

Each $L(\mu)$ is easily seen to be a totally ordered extension of $P(\aleph_\alpha)$. To show that $\{L(\mu): \mu < \omega_\alpha\}$ realizes $P(\aleph_\alpha)$, we need only consider noncomparable elements of $P(\aleph_\alpha)$. Let F_1 and F_2 be distinct noncomparable elements of $P(\aleph_\alpha)$. Since F_1 and F_2 are distinct, there exists a first ordinal $\rho < \omega_\alpha$ such that $F_1'\rho \neq F_2'\rho$, and we may assume that $F_1'\rho < F_2'\rho$ in T_α so that $F_1 < F_2$ in $L(\rho)$. Since F_1 and F_2 are noncomparable in $P(\aleph_\alpha)$, there exists an ordinal $\sigma < \omega_\alpha$ with $\sigma \neq \rho$ for which $F_1'\sigma \geq F_2'\sigma$ in T_α. If $F_1'\sigma > F_2'\sigma$ in T_α, then $F_1 > F_2$ in $L(\sigma)$. If $F_1'\sigma = F_2'\sigma$, then, since ρ is the first ordinal $\nu < \omega_\alpha$ for which $F_1'\nu \neq F_2'\nu$ and $F_1'\rho < F_2'\rho$ in T_α, we again have $F_1 > F_2$ in $L(\sigma)$. Thus $\{L(\mu): \mu < \omega_\alpha\}$ realizes $P(\aleph_\alpha)$, completing the proof of Theorem 3.

Since a totally ordered set is a partially ordered set of dimension 1,

Theorem 3 is a generalization of the theorem of N. Cuesta Dutari [2, p. 243, Theorem 15]. We also note that step (4) of the proof of Theorem 3 is a direct generalization of the corresponding step given by H. Komm [4, pp. 510–511] in his proof of essentially the same theorem for the case $\aleph_\alpha = \aleph_0$.

THEOREM 4. *If* $\lambda = 0$, *or if the General Continuum Hypothesis holds and* λ *is the limit of a denumerable increasing sequence of smaller ordinals, then* P_λ *is an* $\aleph_\lambda$-*universal partially ordered set of cardinality and dimension* $\aleph_\lambda$.

PROOF. (1) P_λ has cardinality and dimension $\aleph_\lambda$ by Theorem 3.

(2) Let P be any partially ordered set of cardinality $\aleph_\lambda$ with an imposed well-ordering, $P = \{p(\mu) : \mu < \omega_\alpha\}$. We define subsets $P(n)$ and P_n of P for each $n < \omega_0$ as follows. If $\lambda = 0$, $P(n) = \{p(n)\}$. If $\lambda > 0$, let $\lambda = \lim \{\lambda(n) : n < \omega_0\}$, where $\lambda(n) < \lambda(n+1)$ for each $n < \omega_0$, so that $\omega_{\lambda(n)} < \omega_{\lambda(n+1)}$ and $\omega_{\lambda(n)+1} < \omega_{\lambda(n+1)+1}$ for each $n < \omega_0$ and $\omega_\lambda = \lim \{\omega_{\lambda(n)} : n < \omega_0\} = \lim \{\omega_{\lambda(n)+1} : n < \omega_0\}$. We set $P(0) = \{p(\mu) : \mu < \omega_{\lambda(0)}\}$; for $0 < n < \omega_0$ we set $P(n) = \{p(\mu) : \omega_{\lambda(n-1)} \leq \mu < \omega_{\lambda(n)}\}$. For every value of λ we set $P_n = \cup \{P(r) : r \leq n\}$; we note that $P = \cup \{P(n) : n < \omega_0\}$.

(3) For each $n < \omega_0$, define $W(n) = \{f : f \in T_\lambda^{P(n)}$ and $\eta[f'p(\mu)] = \mu$ for each $p(\mu) \in P(n)\}$. If $\lambda = 0$, $|W(n)| = 2^n$ and we may write $W(n) = \{f(n, \mu) : \mu < 2^n\}$ with the convention that $f(n, \mu)'p(n) < f(n, \mu+1)'p(n)$ in T_0 for $\mu+1 < 2^n$. If $\lambda > 0$, $|W(n)| = \prod \{2^{|\mu|} : p(\mu) \in P(n)\} = 2^{\langle n \rangle}$, where $\langle n \rangle = \sum \{|\mu| : p(\mu) \in P(n)\} = \aleph_{\lambda(n)}$ so that $|W(n)| = \aleph_{\lambda(n)+1}$; thus we may well-order $W(n)$ and write $W(n) = \{f(n, \mu) : \mu < \omega_{\lambda(n)+1}\}$.

(4) We define functions $g(n, \zeta) : P(n) \to T_\lambda$ for each $n < \omega_0$ and for each $\zeta < \omega_\lambda$ as follows:

(a) Suppose that $\lambda = 0$. We set $g(0, \zeta) = f(0, 0)$ for every $\zeta < \omega_0$. For $0 < n < \omega_0$ and $\zeta < \omega_0$, define $\zeta(n) = \Phi(2^{n(n+1)/2}, \zeta)$ and $\zeta(n, n) = \Theta[2^{n(n-1)/2}, \zeta(n)]$, so that $\zeta(n) < 2^{n(n+1)/2}$ and $\zeta(n) = 2^{n(n-1)/2} \times \zeta(n, n) + r$ with $r < 2^{n(n-1)/2}$ and $\zeta(n, n) < 2^n$; we set $g(n, \zeta) = f[n, \zeta(n, n)]$.

(b) Suppose that $\lambda > 0$. We set $g(0, \zeta) = f[0, \zeta(0)]$ for every $\zeta < \omega_\lambda$, where $\zeta(0) = \Phi(\omega_{\lambda(0)+1}, \zeta)$. For $0 < n < \omega_0$ and $\zeta < \omega_\lambda$, define $\zeta(n) = \Phi(\omega_{\lambda(n)+1}, \zeta)$ and $\zeta(n, n) = \Theta[\omega_{\lambda(n-1)+1}, \zeta(n)]$, so that $\zeta(n) < \omega_{\lambda(n)+1}$ and $\zeta(n) = \omega_{\lambda(n-1)+1} \times \zeta(n, n) + \mu$ with $\mu < \omega_{\lambda(n-1)+1}$ and $\zeta(n, n) < \omega_{\lambda(n)+1}$; we set $g(n, \zeta) = f[n, \zeta(n, n)]$.

(5) We show by induction on n that if $n < \omega_0$ and $G : P_n \to T_\lambda$ is any function such that for each $p(\mu) \in P_n$, $\eta[G'p(\mu)] = \mu$, then there exists an ordinal $\gamma(G) < \omega_\lambda$ such that $G = \cup \{g[r, \gamma(G)] : r \leq n\}$. If $\lambda = 0$, $\gamma(G)$ can be taken $< 2^{n(n+1)/2}$; if $\lambda > 0$, $\gamma(G)$ can be taken $< \omega_{\lambda(n)+1}$.

This statement is clearly true for $n=0$; let $0<n<\omega_0$ and assume that the statement holds for $n-1$. Define $H=G|P_{n-1}$. Then for each $p(\mu)\in P_{n-1}$, $\eta[H'p(\mu)]=\mu$ so that by our inductive assumption, there exists a $\gamma(H)$; if $\lambda=0$, $\gamma(H)<2^{n(n-1)/2}$; if $\lambda>0$, $\gamma(H)<\omega_{\lambda(n-1)+1}$; we have $H=\bigcup\{g[r, \gamma(H)]: r<n\}$. Now $G|P(n)$ is a function $G|P(n)$: $P(n)\to T_\lambda$ such that for each $p(\mu)\in P(n)$, $\eta[G|P(n)'p(\mu)]=\mu$ so that there exists a $\zeta<\omega_\lambda$ for which $G|P(n)=f(n, \zeta)$; if $\lambda=0$, $\zeta<2^n$; if $\lambda>0$, $\zeta<\omega_{\lambda(n)+1}$. We define an ordinal γ as follows: if $\gamma=0$, $\gamma=2^{n(n-1)/2}$ $\times\zeta+\gamma(H)$; if $\lambda>0$, $\gamma=\omega_{\lambda(n-1)+1}\times\zeta+\gamma(H)$. Then for $\lambda=0, \lambda<2^{n(n+1)/2}$, while for $\lambda>0$, $\gamma<\omega_{\lambda(n)+1}$. Clearly, for $r<n$, $g(r, \gamma)=g[r, \gamma(H)]$, while $g(n, \gamma)=f(n, \zeta)$. Thus $G=\bigcup\{g(r, \gamma): r\leqq n\}$ and we may take $\gamma(G)=\gamma$.

(6) Now, depending on the order relations in P, we define functions $k(n, \zeta): P(n)\to T_\lambda$ for every $n<\omega_0$ and every $\zeta<\omega_\lambda$ by induction on n. Let $n<\omega_0$ and $\zeta<\omega_\lambda$ and suppose that we have defined functions $k(r, \zeta): P(r)\to T_\lambda$ for every $r<n$ in such a manner that for each $r<n$ and each $p(\mu)\in P(r)$, $\eta[k(r, \zeta)'p(\mu)]=\mu$, and that $\bigcup\{k(r, \zeta): r<n\}$ generates a totally ordered extension of P_{n-1} (where by convention we take P_{-1} to be the void set). If $\bigcup\{k(r, \zeta): r<n\}\bigcup g(n, \zeta)$ generates a totally ordered extension of P_n we set $k(n, \zeta)=g(n, \zeta)$; otherwise we set $k(n, \zeta)=F[\bigcup\{k(r, \zeta): r<n\}, \Omega(n)]$, the function of Theorem 2 for the present partially ordered set P, where $\Omega(n)=n+1$ if $\lambda=0$ and $\Omega(n)=\omega_{\lambda(n)}$ if $\lambda>0$. By induction, if $\bigcup\{g(r, \zeta): r\leqq n\}$ generates a totally ordered extension of P_n, then $k(r, \zeta)=g(r, \zeta)$ for every $r\leqq n$. Also by induction, for each $\zeta<\omega_\lambda$, $\bigcup\{k(n, \zeta): n<\omega_0\}$ generates a totally ordered extension of P.

(7) We define functions $F(\mu): \bar\omega_\lambda\to T_\lambda$ for each $\mu<\omega_\lambda$ by setting $F(\mu)'\zeta=k(n, \zeta)'p(\mu)$ for each $\zeta<\omega_\lambda$, where n is the integer such that $p(\mu)\in P(n)$. By (4) and (6), each $F(\mu)\in P_\lambda$ and $\Delta[F(\mu)]=\mu$. We define the desired function $F: P\to P_\lambda$ by setting $F'p(\mu)=F(\mu)$ for each $\mu<\omega_\lambda$. By the last remark in (6), we need only check that F preserves the noncomparabilities in P.

(8) Let $n<\omega_0$ and let $G: P_n\to T_\lambda$ be any function which generates a totally ordered extension of P_n in such a manner that for each $p(\mu)\in P_n$, $\eta[G'p(\mu)]=\mu$. By (5), there exists an ordinal $\gamma(G)<\omega_\lambda$ such that $G=\bigcup\{g[r, \gamma(G)]: r\leqq n\}$. But by the next to last sentence of (6), $k[r, \gamma(G)]=g[r, \gamma(G)]$ for $r\leqq n$, so that $G=\bigcup\{k[r, \gamma(G)]: r\leqq n\}$. But now any noncomparability in P can be realized by the extensions $\bigcup\{k[n, \gamma(G_1)]: n<\omega_0\}$ and $\bigcup\{k[n, \gamma(G_2)]: n<\omega_0\}$ of two such functions G_1 and G_2 by Theorem 2 and the theorem of E. Szpilrajn [7]. Thus F is the desired embedding for Theorem 4.

Theorem 4 with $\lambda=0$ has previously been obtained by A. Mostowski [5]. We could, of course, perform a construction analogous to that

in Theorem 4 for any limit cardinal, but we could not make the statement in step (5) of the proof once n had reached ω_0. Step (5) was crucial in proving step (8), which assures us that the function F is actually an embedding. For nonlimit cardinals, Theorem 3 is the best result the author has been able to obtain.

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